Chapter 4 Contravariance, Covariance, and Spacetime Diagrams

4.1 The Components of a Vector in Skewed Coordinates

We have seen in Chapter 3; figure 3.9, that *in order to show inertial motion that is consistent with the Lorentz Transformation, it is necessary to draw coordinate systems that are skewed to each other rather than to use the traditional orthogonal coordinate systems.* It is therefore appropriate to digress for a moment and look into some of the characteristics of a skewed coordinate system. For simplicity and clarity we will start our discussion in two dimensions, later we will extend the discussion to more than two dimensions. Consider the skewed coordinate system shown in figure 4.1. We will use the standard notation that is used in relativity and use superscripts to label the coordinates x^1 and x^2 as shown. (x^2 does not mean "*x* squared", it is just a different means of labeling the coordinates, the reason for which, will become clear in a moment.) A series of lines everywhere parallel to these coordinate axis establishes a space grid. The intersection of any of these two lines establishes a set of coordinates for any particular point considered. Let us now draw the vector **r** in this coordinate system. Now let us find the components of the vector **r** in this skewed coordinate system. But how do we find the components of a vector in a skewed coordinate system?

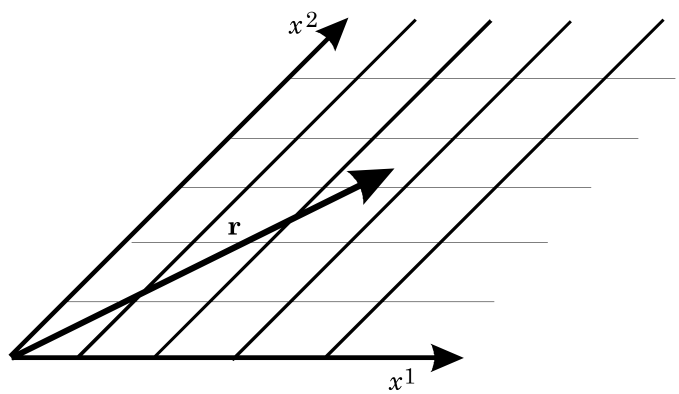

Figure 4.1 A skewed coordinate system.

(a) *Rectangular Components of a Vector in an Orthogonal Coordinate System.*

First, let us recall how we find the components of a vector in an orthogonal coordinate system. To find the *x*-component of the vector we project **r** onto the *x*-axis by dropping a perpendicular line from the tip of **r** down to the *x*-axis as shown in figure 4.2(a). Its intersection with the *x*-axis, we call the *x*-component of the vector. *Note that the line perpendicular to the x-axis*

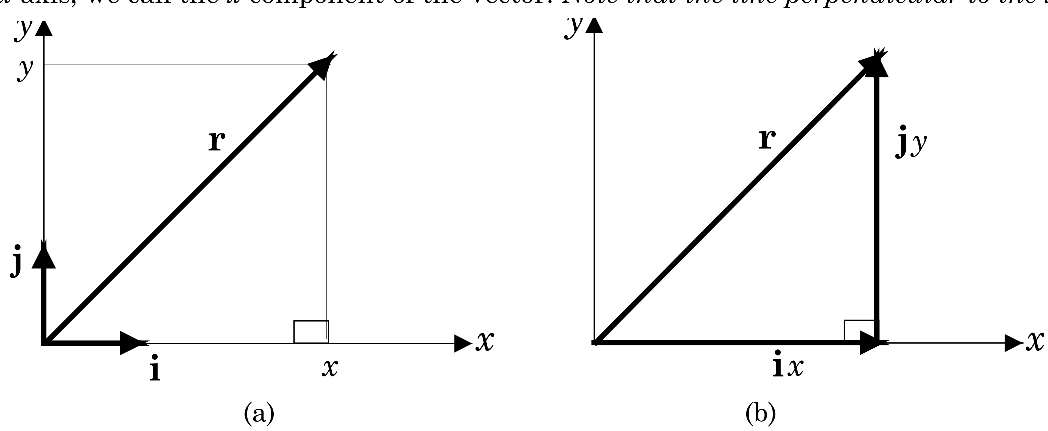

Figure 4.2 An orthogonal coordinate system.

is also parallel to the y-axis. The y-component is found by projecting **r** onto the y-axis by dropping a perpendicular line from the tip of **r** to the y-axis. Its intersection is called the y-component of the vector **r**. *Also note that the line perpendicular to the y-axis is parallel to the x-axis.* In terms of the unit vectors **i** and **j**, and the x and y-components, the vector **r** can be expressed as

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y \tag{4.1}$$

The set of vectors **i** and **j** are sometimes called a set of base vectors.

Implied in the representation of the vector **r** by equation 4.1 is the parallelogram law of vector addition, because $\mathbf{i}x$ is a vector in the x-direction and $\mathbf{j}y$ is a vector in the y-direction. Moving these vectors parallel to themselves generates the parallelogram, and the diagonal of that parallelogram represents the sum of the two vectors $\mathbf{i}x$ and $\mathbf{j}y$ as shown in figure 4.2(b).

Example 4.1

Rectangular components of a vector. A vector **r** has a magnitude of 5 units and makes an angle of 30.0^0 with the x-axis. Find the x- and y-components of this vector.

Solution

The x-component of the vector **r** is found as

$$x = r\cos\theta$$
$$x = 5\cos 30.0^0$$
$$x = 4.33$$

The y-component of the vector **r**, is found as

$$y = r\sin\theta$$
$$y = 5\sin 30.0^0$$
$$y = 2.50$$

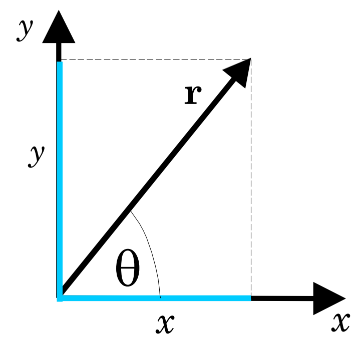

Figure 4.3 The rectangular components of a vector.

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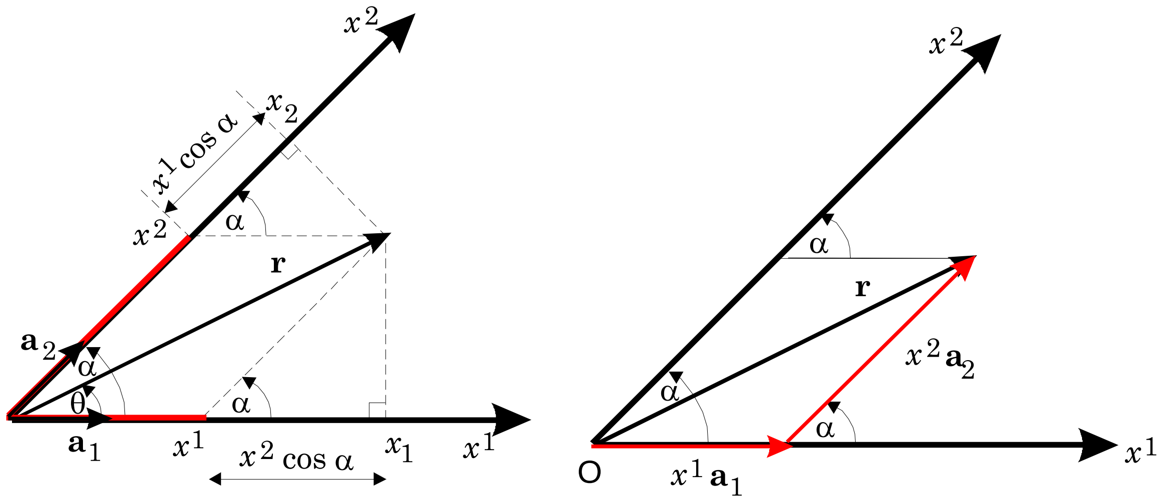
(b) *Contravariant components of a vector.*

Now let us return to the same vector in the skewed coordinate system. We introduce a new system of base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as shown in figure 4.4. The base vector $\mathbf{a}_1$ is in the direction of the x^1 axis and $\mathbf{a}_2$ is in the direction of the x^2 axis. The base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ will be called unitary vectors although they don't necessarily have to be unit vectors. We return to the original question. "How do we find the components of **r**?" For the orthogonal system, the perpendicular from the tip of **r** was perpendicular to one axis and parallel to the other. *For the skewed coordinate system the parallel of one axis is not perpendicular to the other.* ***So there appears to be two ways to find the components of a vector in a skewed coordinate system***.

For the first component let us drop a line from the tip of **r**, *parallel to the x^2 axis,* to the x^1 axis. *This component will be called the contravariant component of the vector **r** and will be designated as x^1 and is shown in red in figure 4.4(a).* Drop another line, *parallel to the x^1 axis*, to the x^2 axis. This gives the second contravariant component x^2, which is also shown in red in figure 4.4(a). In terms of these contravariant components the vector **r** can be written as



(a) Contravariant components (b) Parallelogram law

Figure 4.4. Contravariant components of a vector.

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2 \tag{4.2}$$

We observe from figure 4.4(b) that the vectors $x^1\mathbf{a}_1$ and $x^2\mathbf{a}_2$ add up to the vector $\mathbf{r}$ by the parallelogram law of vector addition. So that equation 4.2 is a valid representation of a vector in the skewed coordinate system.

Example 4.2

Contravariant components of a vector. A vector **r** has a magnitude of 5.00 units and makes an angle of 30.0^0 with the x-axis. If the skewed coordinate system, figure 4.5, makes an angle $\alpha = 70.0^0$, (a) find the contravariant components of this vector, and (b) express the vector in terms of its contravariant components.

Solution

a. The contravariant components of the vector **r** are found from the geometry of figure 4.5. The contravariant component x^1 is found by observing from triangle I

$$\sin\alpha = \frac{r\sin(\alpha - \theta)}{x^1} \tag{4.3}$$

Upon solving for the contravariant component x^1 we get

$$x^1 = \frac{r\sin(\alpha - \theta)}{\sin\alpha} \tag{4.4}$$

$$x^1 = \frac{5.00\sin(70.0^0 - 30.0^0)}{\sin 70^0} = \frac{5.00\,(0.643)}{0.9396} = 5(0.684)$$

$$x^1 = 3.42$$

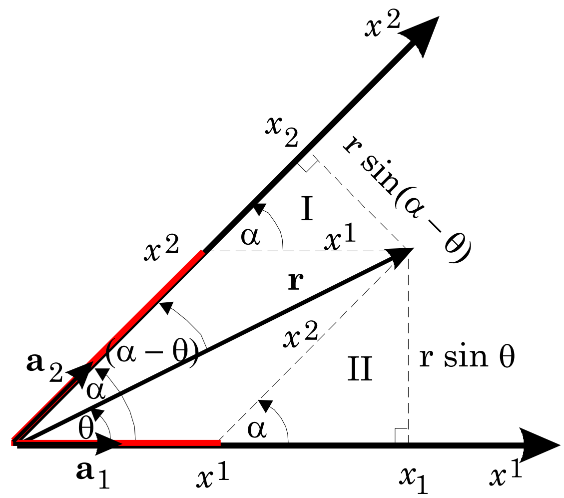


Figure 4.5 Contravariant components.

The contravariant component x^2 is found by observing from triangle II that

$$\sin\alpha = \frac{r\sin\theta}{x^2} \tag{4.5}$$

and upon *solving for the contravariant component x^2 we get*

$$x^2 = \frac{r\sin\theta}{\sin\alpha} \tag{4.6}$$

$$x^2 = \frac{5.00\sin 30^0}{\sin 70^0} = \frac{5.00\,(0.500)}{0.9396} = 5(0.532)$$
$$x^2 = 2.66$$

b. The vector **r** can now be written in terms of its contravariant components from equation 4.2 as

$$\mathbf{r} = 3.42\,\mathbf{a}_1 + 2.66\,\mathbf{a}_2$$

As a check that these are the correct contravariant components of the vector **r**, let us determine the magnitude r from this result. We can no longer use the Pythagorean Theorem to determine r, because we no longer have a right triangle as we do in the case of rectangular components. We can however apply the law of cosines to the triangle of figure 4.4(b) to obtain

$$r^2 = (x^1)^2 + (x^2)^2 + 2x^1x^2\cos\alpha$$
$$= (3.42)^2 + (2.66)^2 + 2(3.42)(2.66)\cos(70.0^0)$$
$$= (11.7) + (7.08) + (18.2)\,(0.342)$$
$$= 18.8 + 6.23 = 25.0$$
$$r = 5.00$$

We see that we do get the correct result for the magnitude of the vector **r**.

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(c) *Covariant components of a vector.*

For a second representation of the components of a vector in a skewed coordinate system, we now *drop a perpendicular from the tip of* ***r*** *to the x^1 axis* intersecting it at the point that we will now designate as x_1, and is shown in yellow in figure 4.6(a). *We will call x_1 a covariant component of the vector* ***r***. Now *drop a perpendicular from the tip of* ***r*** *to the x^2 axis*, *obtaining the second covariant component x_2,* also shown as a yellow line. We now have the two vector components $x_1\mathbf{a}_1$ and $x_2\mathbf{a}_2$. But these vector components do not satisfy the parallelogram law of vector addition when we try to add them together head to tail, as is obvious from figure 4.6(b). That is, by adding the vectors from head to tail, you can see that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ will not add up to the vector **r**. In fact you can see that the sum would be actually greater than the magnitude of **r**, and would not be in the correct direction. Therefore in terms of these components

$$\mathbf{r} \neq x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2$$

At first glance it therefore seems that the only way we can find the components of a vector that is consistent with the parallelogram law of vector addition is to drop lines from the tip of **r** that are parallel to the coordinate axis, thereby obtaining the contravariant components of a vector.

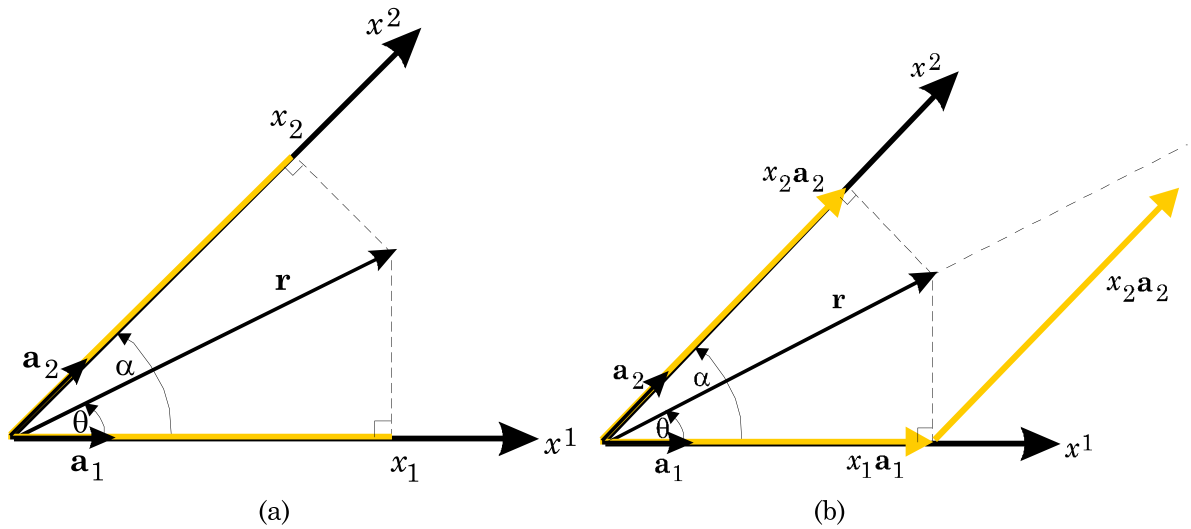

Figure 4.6 Covariant components of a vector.

However there is still another way to determine the components of the vector **r** and that is to establish a new coordinate system with unit vectors $\mathbf{e}^1$ and $\mathbf{e}^2$ where $\mathbf{e}^1$ is perpendicular to $\mathbf{a}_2$ and $\mathbf{e}^2$ is perpendicular to $\mathbf{a}_1$. This new base system is shown in figure 4.7(a), along with the old base system. The base vector $\mathbf{e}^1$ defines the direction of a new axis x_1, while the base vector $\mathbf{e}^2$ defines a

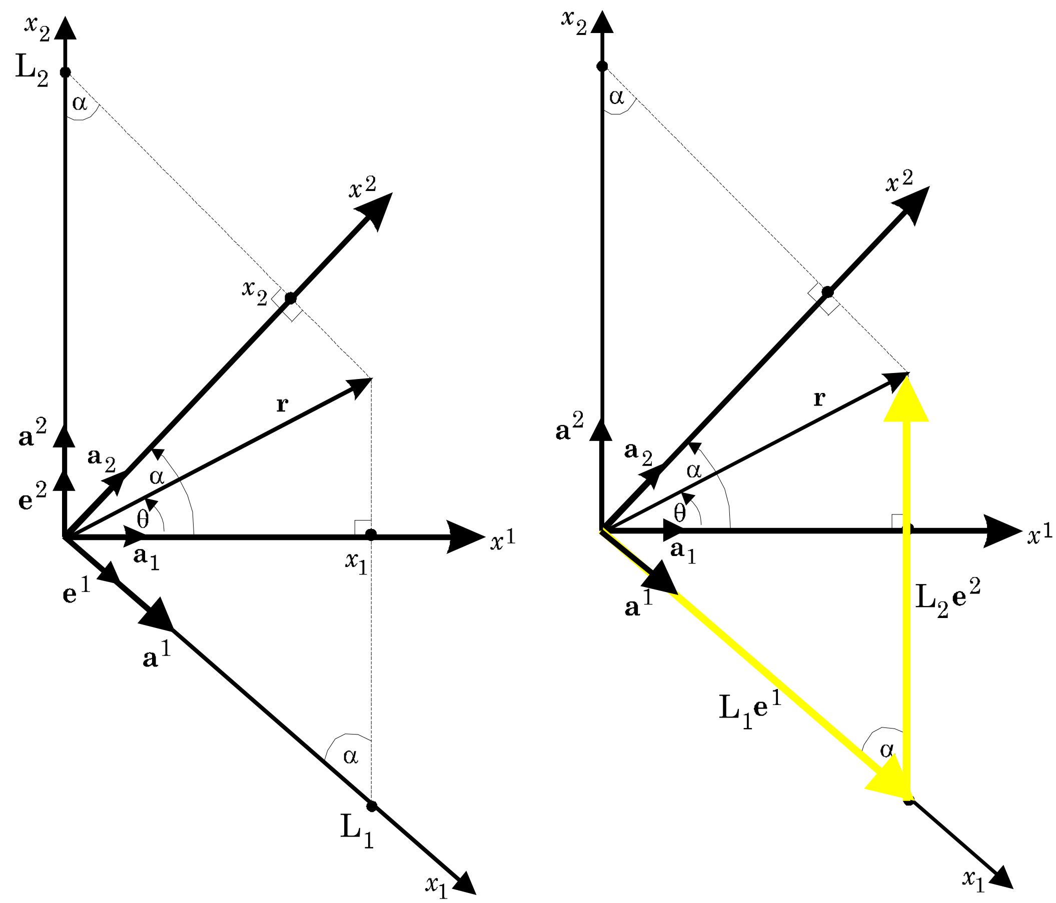

Figure 4.7 Introduction of some new base vectors.

new axis x_2. We now drop a perpendicular from the tip of **r** to the x^1 axis, but instead of terminating the perpendicular at the x^1 axis, we continue it down until it intersects the new x_1 axis. (Note that the perpendicular line is perpendicular to the x^1-axis but not the x_1 axis.) We call the projection on the new axis, L_1. Now drop a perpendicular from the tip of **r** to x^2, then extrapolate it until it crosses the new x_2 axis at the point L_2. Then we can see from figure 4.7, that the vectors $L_1\mathbf{e}^1$ and $L_2\mathbf{e}^2$ will add up by the parallelogram law of vector addition to

$$\mathbf{r} = L_1\mathbf{e}^1 + L_2\mathbf{e}^2 \tag{4.7}$$

Can we express the lengths L_1 and L_2, in terms of the covariant components x_1 and x_2? Referring back to figure 4.4(a), we first note that the angle α is the measure angle of the amount of skewness of the coordinates. The *covariant component* x_1 can be seen to be composed of two lengths, i.e.

$$x_1 = x^1 + x^2 \cos\alpha \tag{4.8}$$

while the *covariant component* x_2 is composed of the two lengths

$$x_2 = x^2 + x^1 \cos\alpha \tag{4.9}$$

From the bottom triangle in figure 4.7(a) we observe that

$$\sin\alpha = \frac{x_1}{L_1}$$

and hence

$$L_1 = \frac{x_1}{\sin\alpha} \tag{4.10}$$

And from the upper triangle in figure 4.7(a) we have

$$\sin\alpha = \frac{x_2}{L_2}$$

Therefore

$$L_2 = \frac{x_2}{\sin\alpha} \tag{4.11}$$

Replacing equations 4.10 and 4.11 into equation 4.7 gives

$$\mathbf{r} = \frac{x_1}{\sin\alpha}\mathbf{e}^1 + \frac{x_2}{\sin\alpha}\mathbf{e}^2 \tag{4.12}$$

or upon slightly rearranging terms, we can write this as

$$\mathbf{r} = x_1 \frac{\mathbf{e}^1}{\sin\alpha} + x_2 \frac{\mathbf{e}^2}{\sin\alpha}$$

If we now define the two new base vectors

$$a^1 = \frac{e^1}{\sin\alpha} \tag{4.13}$$

and

$$a^2 = \frac{e^2}{\sin\alpha} \tag{4.14}$$

then the vector **r** can be written in terms of the covariant components as

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 \tag{4.15}$$

Since equation 4.15 is just equation 4.7, but in different notation, it also satisfies the parallelogram law of vector addition.

Example 4.3

Covariant components of a vector. A vector **r** has a magnitude of 5.00 units and makes an angle of 30.0^0 with the x-axis. If the skewed coordinate system makes an angle $\alpha = 70.0^0$, (a) find the covariant components of this vector, (b) express the vector in terms of its covariant components, and (c) find the values of the base vectors.

Solution

a. To express the vector in terms of its covariant components we use equation 4.15

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2$$

The covariant component x_1 is found from figure 4.7(a) as

$$x_1 = r\cos\theta \tag{4.16}$$
$$x_1 = 5.00\cos(30.0^0) = 5.00\,(0.866)$$
$$x_1 = 4.33$$

while the covariant component x_2 is found from figure 4.7(a) as

$$x_2 = r\cos(\alpha - \theta) \tag{4.17}$$
$$x_2 = 5.00\cos(70.0^0 - 30.0^0) = 5.00\cos(40.0^0)$$
$$x_2 = 3.83$$

As a check let us find the covariant component x_1 from equation 4.8 in terms of the contravariant components as

$$x_1 = x^1 + x^2\cos\alpha \tag{4.8}$$

The values of x^1 and x^2 were determined in Example 4.2 to be

$$x^1 = \frac{r\sin(\alpha - \theta)}{\sin\alpha} \tag{4.4}$$
$$x^1 = \frac{5.00\sin(70.0^0 - 30.0^0)}{\sin 70^0} = 3.42$$

and

$$x^2 = \frac{r\sin\theta}{\sin\alpha} \tag{4.6}$$
$$x^2 = \frac{5.00\sin 30^0}{\sin 70^0} = 2.66$$

Replacing these values into equation 4.8 gives for the covariant component x_1

$$x_1 = x^1 + x^2\cos\alpha \tag{4.8}$$
$$x_1 = 3.42 + 2.66\cos 70.0^0 = 3.42 + 0.910$$

$$x_1 = 4.33$$

The covariant component x_2 is found from equation 4.9 in terms of the contravariant components as

$$x_2 = x^2 + x^1 \cos\alpha \tag{4.9}$$
$$x_2 = 2.66 + 3.42 \cos 70.0^0 = 2.66 + 1.17$$
$$x_2 = 3.83$$

Notice that the components are the same for either procedure.

b. The vector **r** in terms of the covariant components is obtained from equation 4.15 as

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2$$

Hence,

$$\mathbf{r} = 4.33\ \mathbf{a}^1 + 3.83\ \mathbf{a}^2$$

As a check that these are the correct covariant components of the vector r, let us determine the magnitude r from this result. We can no longer use the Pythagorean Theorem to determine r, because we no longer have a right triangle as we do in the case of rectangular components. We can however apply the law of cosines to the triangle of figure 4.4(b) to obtain

$$r^2 = (x_1)^2 + (x_2)^2 - 2x_1x_2 \cos \alpha$$
$$r^2 = (4.33)^2 + (3.83)^2 - 2(4.33)(3.83) \cos 70$$
$$r = 4.70 \text{ units}$$

But something is wrong here. We know the magnitude of r should be 5 and it is not. The trouble is that the unitary vectors are not unit vectors but unitary vectors. They are not equal to one. We have to take these vectors into account. They are taken into account by using the base vectors $\mathbf{e}^1$ and $\mathbf{e}^2$ which are unit vectors.

c. The base vector $\mathbf{a}^1$ is found from equation 4.13 as

$$\mathbf{a}^1 = \frac{\mathbf{e}^1}{\sin\alpha} \tag{4.13}$$
$$\mathbf{a}^1 = \frac{\mathbf{e}^1}{\sin\alpha} = \frac{\mathbf{e}^1}{\sin 70.0^0}$$
$$\mathbf{a}^1 = 1.06\ \mathbf{e}^1$$

The base vector $\mathbf{a}^2$ is found from equation 4.14 as

$$\mathbf{a}^2 = \frac{\mathbf{e}^2}{\sin\alpha} \tag{4.14}$$
$$\mathbf{a}^2 = \frac{\mathbf{e}^2}{\sin\alpha} = \frac{\mathbf{e}^2}{\sin 70.0^0}$$
$$\mathbf{a}^2 = 1.06\ \mathbf{e}^2$$

Notice that the base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$ are not unit vectors, and their value will vary depending upon the angle α that the coordinates are skewed. If we wish we could also write the vector in terms of the unit vectors $\mathbf{e}^1$ and $\mathbf{e}^2$ by using equations 4.13 and 4.14.

$$r = x_1a^1 + x_2a^2$$
$$\mathbf{r} = 4.33\ \mathbf{a}^1 + 3.83\ \mathbf{a}^2$$
$$\mathbf{r} = (4.33)(1.06)\mathbf{e}^1 + (3.83)(1.06)\mathbf{e}^2$$

$$\mathbf{r} = (4.61)\mathbf{e}^1 + (4.08)\mathbf{e}^2$$

Recall from equation 4.7 and figure 4.7b that

$$\mathbf{r} = L_1\mathbf{e}^1 + L_2\mathbf{e}^2 \tag{4.7}$$

where $L_1 = x_1$ times the magnitude of the unitary vector, $\mathbf{a}^1$. That is

$$L_1 = x_1\ \mathbf{a}^1$$
$$L_1 = (4.33)(1.06) = 4.61$$

and

$$L_2 = (3.83)(1.06) = 4.08$$

As a check that these are the correct covariant components of the vector **r**, let us determine the magnitude r from this result. As we showed earlier for the contravariant vector, we can no longer use the Pythagorean Theorem to determine r, because we no longer have a right triangle as we do in the case of rectangular components. We can however apply the law of cosines to the triangle of figure 4.7(b) to obtain

$$r^2 = (L_1)^2 + (L_2)^2 - 2L_1L_2 \cos\alpha$$
$$r^2 = (4.61)^2 + (4.08)^2 - 2(4.61)(4.08)\cos 70$$
$$r^2 = 25.0$$
$$r = 5.00 \text{ units}$$

Notice that we get the same magnitude of 5 units as we did in examples 4.1 and 4.2.

We could also use equation 4.7 by first determining the values of L_1 and L_2 from the equations

$$L_1 = x_1 / \sin\alpha$$
$$L_1 = (4.33) / \sin(70)$$
$$L_1 = (4.61)$$

and

$$L_2 = x_2 / \sin\alpha$$
$$L_2 = (3.83) / \sin(70)$$
$$L_2 = (4.08)$$

and placing these into the law of cosines we get

$$r^2 = (L_1)^2 + (L_2)^2 - 2L_1L_2 \cos\alpha$$
$$r^2 = (4.61)^2 + (4.08)^2 - 2(4.61)(4.08)\cos 70$$
$$r^2 = 25.0$$
$$r = 5.00 \text{ units}$$

Again notice that we get the same correct result for the magnitude of the vector r.

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To summarize our results, *equation 4.2 is the representation of the vector* ***r*** *in terms of its contravariant components,*

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2 \tag{4.2}$$

while *equation 4.15 is the representation of the vector* ***r*** *in terms of its covariant components*, i.e.

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 \tag{4.15}$$

So in a skewed coordinate system there are two types of components- contravariant and covariant. ***The contravariant components are found by parallel projections onto the coordinate axes while the covariant components are found by perpendicular projections*.** ***Contravariant components are designated by superscripts,*** x^i, ***while covariant components are designated by subscripts,*** x_i**.** The base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$ are not unit vectors even if $\mathbf{a}_1$ and $\mathbf{a}_2$ are.

The distinction between contravariant and covariant components disappears in orthogonal coordinates, because the axes are orthogonal. That is, in orthogonal coordinates, a projection which is parallel to one axis, is also perpendicular to the other. Let us now return to the spacetime diagrams we discussed in chapter 2 and see how these concepts of covariance and contravariance are applied to these spacetime diagrams.

4.2 Different Forms of The Spacetime Diagrams

Figure 2.9 showed the relation between the *S* and *S'* frames of reference in spacetime. The *S* frame was the stationary frame and *S'* was a frame moving to the right with the velocity *v*. The angle θ, of figure 2.9 was given by equation 2.3 as

$$\theta = \tan^{-1}\frac{v}{c}$$

But we already said that there is no frame of reference that is absolutely at rest, and yet our diagram shows the preferred stationary frame, *S*, as an orthogonal coordinate system while the moving frame, *S'*, is an acute skewed coordinate system. So it seems as if the rest frame is a special frame compared to the moving frame. However, the principle of relativity says that all frames are equivalent. That is, there should be no distinction between a frame of reference that is at rest or one that is moving at a constant velocity *v*. Figure 3.9 should be modified to show that there is no preferential frame of reference. We showed in Chapter 2, that if body 1 is at rest and body 2 moves to the right with a velocity **v**, that this is equivalent to body 2 being at rest and body 1 moving to the left with the velocity –**v**. Another equivalence is to have an arbitrary observer at rest between 1 and 2 and body 2 can move to the right with a velocity **v**/2 with respect to the frame at rest and body 1 can move to the left with a velocity –**v**/2.

We can incorporate these generalities by redrawing figure 3.9 for *S'* with the τ' axis now making an angle θ/2 with the original τ-axis, and by showing a second observer, *S"*, moving to the left of the stationary observer with the velocity –**v**/2. This is shown in the spacetime diagram of figure 4.8 as the τ" axis making an angle –θ/2 with the τ-axis. The angle θ/2 is computed in the same way as the computation for the τ'-axis, that is,

$$\theta/2 = \tan^{-1}\frac{v/2}{c}$$ *S'* frame moving to the right with velocity **v**/2 with respect to *S* frame

$$-\theta/2 = \tan^{-1}\frac{-v/2}{c}$$ *S"* frame moving to the left with velocity –**v**/2 with respect to *S* frame

In this way the *S'* frame will be moving at the velocity **v** with respect to the *S"* frame. Similarly an *x*"-axis can be drawn at an angle θ/2 from the –*x*-axis. Notice that the *x*" and τ" are found in the same way that we found *x*' and τ', except that *x*" and τ" have negative slopes, indicative of the motion to the left. These new *x*"- and τ"-axes generate a new acute skewed coordinate system, *S"*, located in the fourth quadrant, as seen in figure 4.8(a) and 4.8(b). Note that the *S'* coordinate system is shown in blue while the *S"* coordinate system is shown in red. Also notice that because of the symmetry, the scales are the same in the *S"* frame as they are in the *S'* frame, which is of course different to the scale in the *S* frame as we showed before. Also note that in figure 2.9, θ was the angle between τ' and τ because *S'* was moving at the speed *v* with respect to the *S* frame. Now notice that θ/2 is the angle between τ' and τ because *S'* is now moving at the speed *v*/2 with respect to the *S* frame of reference. Also note that θ is now the angle between τ' and τ" because *S'* is now moving at the speed *v* with respect to *S"*.

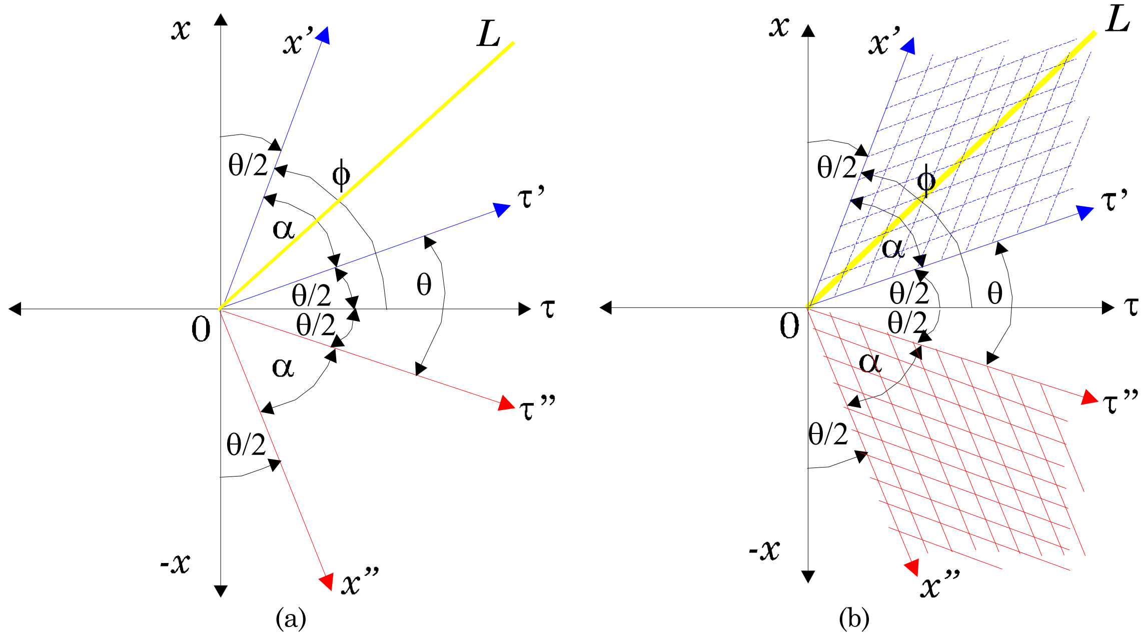


Figure 4.8 Relation of *S'* and *S"* frame of references.

Also note from figure 4.8 that *the x"-axis is orthogonal to the τ'-axis* since

$$\angle\, \tau' 0 x'' = \alpha + \theta$$

but $\alpha = 90^0 - \theta$. Hence,

$$\angle\, \tau' 0 x'' = 90^0 - \theta + \theta$$

and

$$\angle\, \tau' 0 x'' = 90^0$$

Similarly, the τ"*-axis is perpendicular to the x'-axis*, since

$$\angle\, x' 0 \tau'' = \phi + \theta/2$$

but $\phi = 90^0 - \theta/2$. Hence,

$$\angle\, x' 0 \tau'' = 90^0 - \theta/2 + \theta/2$$

and

$$\angle\, x' 0 \tau'' = 90^0$$

The fact that the *x*"-axis is orthogonal to the τ'-axis, and the τ"-axis is orthogonal to the *x*'-axis, should remind us of how the x_1 axis was perpendicular to the x^2 axis and the x^1 axis was perpendicular to the x_2 axis in figure 4.7 in our study of some of the characteristics of covariant and contravariant components. We will return to this similarity shortly.

Figure 4.8 shows that the *S'* and *S"* frames are symmetrical with respect to the *S* frame of reference, but not with respect to each other. Both frames should also measure the same velocity of light *c*, which is assured if the light line *OL* were to bisect both sets of coordinate axes. Also note that because of the symmetry of both *S'* and *S"* frames, they would both intersect the scale hyperbolas at the same values. Hence, the scale on the *S'* frame is the same as the scale on the *S"* frame. We can modify figure 4.8 by reflecting the *x*"-axis in the fourth quadrant, through the origin of the coordinates, to make an *x*"-axis in the second quadrant, as shown in figure 4.9. Note that now the light line *OL* does indeed bisect the *x*',τ'-axes and the *x*",τ"-axes guaranteeing that the speed of light is same in both coordinate systems. The *S"* coordinate system is now an obtuse skewed coordinate system instead of the acute one it was in the fourth quadrant. Figure 4.9 should now be used to describe events in the *S'* and *S"* coordinate systems, instead of figure 3.9 which described events in the *S* and *S'* frames of reference.

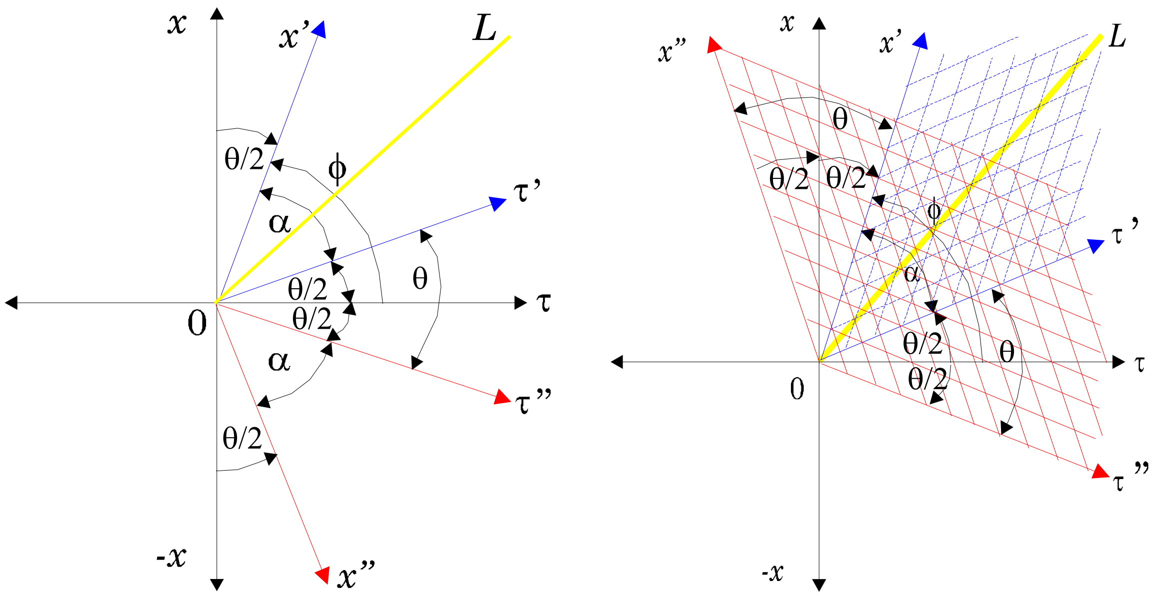


Figure 4.9 New *S'* and *S"* frame of references.

4.3 Reciprocal Systems of Vectors

We have discussed the spacetime diagrams in two dimensions. We would like to extend that discussion first into three dimensions and then into four or more dimensions. In order to extend this discussion we must first discuss the concept of reciprocal systems of vectors. Consider the three dimensional oblique coordinate system shown in figure 4.31. The three axes are described by the constant unitary vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ as shown in the figure. We now define the set of reciprocal unitary vectors as

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \tag{4.16}$$

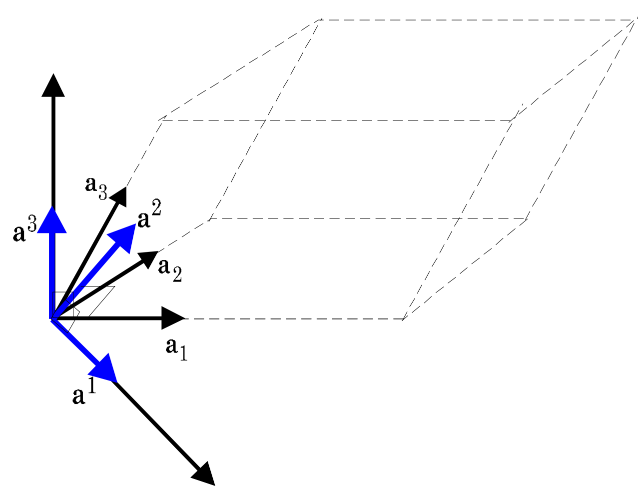


Figure 4.31 The reciprocal unitary vectors.

$$\mathbf{a}^2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \tag{4.17}$$

$$\mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \tag{4.18}$$

By nature of the cross product of two vectors, and as can be seen in figure 4.31, $\mathbf{a}^1$ is perpendicular to the plane generated by $\mathbf{a}_2$ and $\mathbf{a}_3$; $\mathbf{a}^2$ is perpendicular to the plane generated by $\mathbf{a}_3$ and $\mathbf{a}_1$; and $\mathbf{a}^3$ is perpendicular to the plane generated by $\mathbf{a}_1$ and $\mathbf{a}_2$. Hence, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are called unitary vectors, while the vectors $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ are called reciprocal unitary vectors.

Let us now consider combinations of products of these unitary vectors and their reciprocal unitary vectors. First, let us consider the product

$$\mathbf{a}^1 \bullet \mathbf{a}_1 = \left(\frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right) \bullet \mathbf{a}_1 = \frac{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} = 1 \tag{4.19}$$

and

$$\mathbf{a}^2 \bullet \mathbf{a}_2 = \left(\frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right) \bullet \mathbf{a}_2 = \frac{\mathbf{a}_2 \bullet (\mathbf{a}_3 \times \mathbf{a}_1)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \tag{4.20}$$

But as you recall from vector analysis, by the vector triple product of three vectors, a cyclic interchange of letters is permissible, that is,

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$$

Applying this to our unitary vectors we get

$$\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3) = \mathbf{a}_2 \bullet (\mathbf{a}_3 \times \mathbf{a}_1) = \mathbf{a}_3 \bullet (\mathbf{a}_1 \times \mathbf{a}_2) \tag{4.21}$$

Using equation 4.21 in equation 4.20 gives

$$\mathbf{a}^2 \bullet \mathbf{a}_2 = \left(\frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right) \bullet \mathbf{a}_2 = \frac{\mathbf{a}_2 \bullet (\mathbf{a}_3 \times \mathbf{a}_1)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} = \frac{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} = 1 \tag{4.22}$$

Again using equation 4.18 we find for the third product

$$\mathbf{a}^3 \bullet \mathbf{a}_3 = \left(\frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right) \bullet \mathbf{a}_3 = \frac{\mathbf{a}_3 \bullet (\mathbf{a}_1 \times \mathbf{a}_2)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} = \frac{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} = 1 \tag{4.23}$$

The results of equations 4.19, 4.22, and 4.23 shows that the product of a unitary vector and its reciprocal unitary vector is equal to one, that is,

$$\mathbf{a}^1 \bullet \mathbf{a}_1 = \mathbf{a}^2 \bullet \mathbf{a}_2 = \mathbf{a}^3 \bullet \mathbf{a}_3 = 1 \tag{4.24}$$

When we consider the mixed products of these unitary vectors and their reciprocal unitary vectors we get

$$\mathbf{a}^2 \bullet \mathbf{a}_1 = \left(\frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right) \bullet \mathbf{a}_1 = \frac{\mathbf{a}_1 \bullet (\mathbf{a}_3 \times \mathbf{a}_1)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}$$

But as can be seen in figure 4.31, $\mathbf{a}_3 \times \mathbf{a}_1$ is perpendicular to the plane generated by $\mathbf{a}_1$ and $\mathbf{a}_3$ and hence is perpendicular to the vector $\mathbf{a}_1$ and hence its dot product with $\mathbf{a}_1$ is equal to zero. That is,

$$|\mathbf{a}_1 \bullet (\mathbf{a}_3 \times \mathbf{a}_1)| = |\mathbf{a}_1|\,|\mathbf{a}_3 \times \mathbf{a}_1| \cos 90^0 = 0$$

Hence

$$\mathbf{a}^2 \bullet \mathbf{a}_1 = 0$$

Similarly the mixed product

$$\mathbf{a}^3 \bullet \mathbf{a}_1 = \left(\frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \right) \bullet \mathbf{a}_1 = \frac{\mathbf{a}_1 \bullet (\mathbf{a}_1 \times \mathbf{a}_2)}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}$$

But as can also be seen in figure 4.31, $\mathbf{a}_1 \times \mathbf{a}_2$ is perpendicular to the plane generated by $\mathbf{a}_1$ and $\mathbf{a}_2$ and hence is perpendicular to the vector $\mathbf{a}_1$ and hence its dot product with $\mathbf{a}_1$ is equal to zero. That is,

$$|\mathbf{a}_1 \bullet (\mathbf{a}_1 \times \mathbf{a}_2)| = |\mathbf{a}_1||\mathbf{a}_1 \times \mathbf{a}_2| \cos 90^0 = 0$$

Hence the dot product of $\mathbf{a}_1 \bullet (\mathbf{a}_1 \times \mathbf{a}_2)$ is equal to zero, therefore

$$\mathbf{a}^3 \bullet \mathbf{a}_1 = 0$$

In a similar way, all the mixed products of the unitary vectors and the reciprocal unitary vectors are equal to zero. That is,

$$\mathbf{a}^1 \bullet \mathbf{a}_2 = \mathbf{a}^1 \bullet \mathbf{a}_3 = \mathbf{a}^2 \bullet \mathbf{a}_1 = \mathbf{a}^2 \bullet \mathbf{a}_3 = \mathbf{a}^3 \bullet \mathbf{a}_1 = \mathbf{a}^3 \bullet \mathbf{a}_2 = 0 \tag{4.25}$$

In summary, the reciprocal unitary vectors are defined by equations 4.16, 4.17, and 4.18, and the product of these unitary vectors and the reciprocal unitary vectors are summarized in equations 4.24 and 4.25.

Just as equations 4.16 - 4.18 expressed the reciprocal unitary vectors in terms of the unitary vectors, the unitary vectors can be expressed in terms of the reciprocal unitary vectors by the same reciprocal relations. That is,

$$\mathbf{a}_1 = \frac{\mathbf{a}^2 \times \mathbf{a}^3}{\mathbf{a}^1 \bullet (\mathbf{a}^2 \times \mathbf{a}^3)} \tag{4.26}$$

$$\mathbf{a}_2 = \frac{\mathbf{a}^3 \times \mathbf{a}^1}{\mathbf{a}^1 \bullet (\mathbf{a}^2 \times \mathbf{a}^3)} \tag{4.27}$$

$$\mathbf{a}_3 = \frac{\mathbf{a}^1 \times \mathbf{a}^2}{\mathbf{a}^1 \bullet (\mathbf{a}^2 \times \mathbf{a}^3)} \tag{4.28}$$

Since the order of a dot product is not significant, that is, $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$, the combinations of all the products in equation 4.19 through 4.25 are the same. That is,

$$\mathbf{a}^1 \bullet \mathbf{a}_1 = \mathbf{a}_1 \bullet \mathbf{a}^1 = \mathbf{a}^2 \bullet \mathbf{a}_2 = \mathbf{a}_2 \bullet \mathbf{a}^2 = \mathbf{a}^3 \bullet \mathbf{a}_3 = \mathbf{a}_3 \bullet \mathbf{a}^3 = 1 \tag{4.29}$$

and

$$\begin{aligned} \mathbf{a}^1 \bullet \mathbf{a}_2 &= \mathbf{a}_1 \bullet \mathbf{a}^2 = \mathbf{a}^1 \bullet \mathbf{a}_3 = \mathbf{a}_1 \bullet \mathbf{a}^3 \\ &= \mathbf{a}^2 \bullet \mathbf{a}_1 = \mathbf{a}_2 \bullet \mathbf{a}^1 = \mathbf{a}^2 \bullet \mathbf{a}_3 = \mathbf{a}_2 \bullet \mathbf{a}^3 \\ &= \mathbf{a}^3 \bullet \mathbf{a}_1 = \mathbf{a}_3 \bullet \mathbf{a}^1 = \mathbf{a}^3 \bullet \mathbf{a}_2 = \mathbf{a}_3 \bullet \mathbf{a}^2 = 0 \end{aligned} \tag{4.30}$$

If we apply the same reasoning process to the orthogonal **i**, **j**, **k**, system of unit vectors we get

$$\mathbf{i}^1 = \frac{\mathbf{j} \times \mathbf{k}}{\mathbf{i} \bullet (\mathbf{j} \times \mathbf{k})} = \frac{\mathbf{i}}{\mathbf{i} \bullet \mathbf{i}} = \mathbf{i}$$

$$\mathbf{j}^1 = \frac{\mathbf{k} \times \mathbf{i}}{\mathbf{i} \bullet (\mathbf{j} \times \mathbf{k})} = \frac{\mathbf{j}}{\mathbf{i} \bullet \mathbf{i}} = \mathbf{j}$$

$$\mathbf{k}^1 = \frac{\mathbf{i} \times \mathbf{j}}{\mathbf{i} \bullet (\mathbf{j} \times \mathbf{k})} = \frac{\mathbf{k}}{\mathbf{i} \bullet \mathbf{i}} = \mathbf{k}$$

Therefore, the reciprocal vectors of **i**, **j**, **k**, are the vectors **i**, **j**, **k**, themselves. In fact for any orthogonal set of unit vectors, whether rectangular, spherical, cylindrical etc., the reciprocal unit vectors will be the unit vectors themselves. All orthogonal sets of vectors are self-reciprocal. *The only time we will have reciprocal sets of vectors is when we have oblique coordinate systems, as we do in our spacetime diagrams.*

In section 4.1 we analyzed a skewed coordinate system in two-dimensions and showed that we could represent a vector in that two-dimensional system by using either contravariant or covariant components of a vector. *That is, we found the vector **r** could be written in terms of the contravariant components x^1 and x^2 as*

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2 \tag{4.2}$$

and in terms of the covariant components x_1 and x_2 as

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 \tag{4.15}$$

Remember that the contravariant components were found by dropping lines that were parallel to the appropriate axes, while the covariant components were found by dropping lines that were perpendicular to the appropriate axes.

*Now that we have established a three dimensional skewed coordinate system, we can now write the vector **r**, in figure 4.32, in terms of the three dimensional contravariant components x^1, x^2, and x^3, and the unitary vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$* as

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2 + x^3\mathbf{a}_3 \tag{4.31}$$

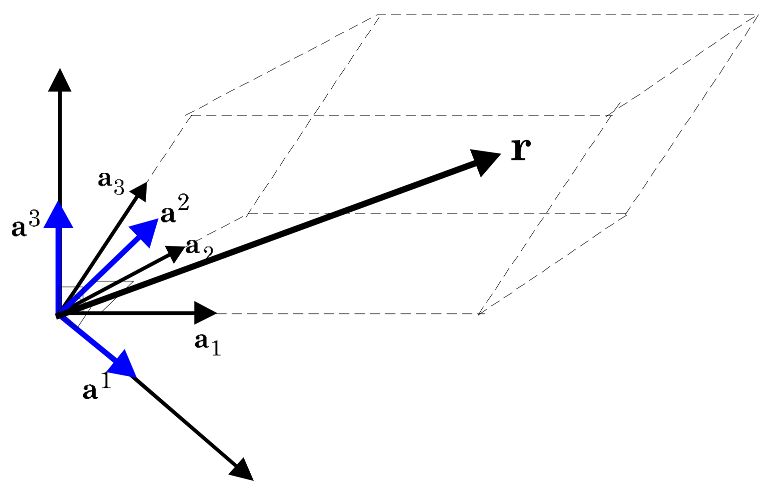


Figure 4.32 Three dimensional skewed coordinate system.

*The vector **r** can also be expressed in terms of the covariant components x_1, x_2, and x_3 of the vector and the reciprocal system of unitary vectors $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ as*

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 + x_3\mathbf{a}^3 \tag{4.32}$$

where the reciprocal unit vectors $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ are given by equations 4.16, 4.17, and 4.18.

Example 4.4

In section 4.1 we showed that we could establish a new coordinate system with unit vectors $\mathbf{e}^1$ and $\mathbf{e}^2$ where $\mathbf{e}^1$ is perpendicular to $\mathbf{a}_2$ and $\mathbf{e}^2$ is perpendicular to $\mathbf{a}_1$. The base vector $\mathbf{e}^1$ defined the direction of a new axis x_1, while the base vector $\mathbf{e}^2$ defined a new axis x_2. In this new set of coordinate we showed that a vector $\mathbf{r}$ could be written in terms of the covariant components as

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 \tag{4.15}$$

if we defined the two new base vectors

$$\mathbf{a}^1 = \frac{\mathbf{e}^1}{\sin\alpha} \tag{4.13}$$

and

$$\mathbf{a}^2 = \frac{\mathbf{e}^2}{\sin\alpha} \tag{4.14}$$

Show that equation 4.13 is equivalent to equation 4.16

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \tag{4.16}$$

Solution

The reciprocal unitary vector $\mathbf{a}^1$ is given by

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}$$

But the angle between the unitary vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ is the skew angle α of the coordinates. Hence,

$$\mathbf{a}_2 \times \mathbf{a}_3 = |\mathbf{a}_2||\mathbf{a}_3| \sin\alpha = \sin\alpha$$

Where $|\mathbf{a}_2| = |\mathbf{a}_3| = 1$ since they are unit vectors. Also since the angle between the vectors $\mathbf{a}_1$ and $(\mathbf{a}_2 \times \mathbf{a}_3)$ is $(90^0 - \alpha)$, then

$$\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3) = |\mathbf{a}_1||(\mathbf{a}_2 \times \mathbf{a}_3)| \cos(90^0 - \alpha)$$
$$= |\mathbf{a}_1||\mathbf{a}_2||\mathbf{a}_3| \sin\alpha \cos(90^0 - \alpha)$$

But $|\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = 1$ since they are unit vectors. Therefore

$$\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3) = \sin\alpha \cos(90^0 - \alpha)$$

However,

$$\cos(90^0 - \alpha) = \cos 90^0 \cos\alpha + \sin 90^0 \sin\alpha = \sin\alpha$$

Therefore

$$\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3) = \sin\alpha \cos(90^0 - \alpha) = \sin^2\alpha$$

Replacing these values in equation 4.16 gives

$$|\mathbf{a}^1| = \left|\frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)}\right| = \frac{\sin\alpha}{\sin^2\alpha} = \frac{1}{\sin\alpha}$$

and hence

$$\mathbf{a}^1 = \frac{\mathbf{e}^1}{\sin\alpha}$$

Hence the case shown in section 4.1 for finding the covariant components of a vector is a special case of the reciprocal system of vectors.

4.4 Example of The Use of Covariant and Contravariant Vectors

As we have seen, any vector can be written in two ways. One in terms of the contravariant components and the other in terms of the covariant components. *A vector written in terms of its contravariant components is called a contravariant vector. A vector written in terms of its covariant components is called a covariant vector.* As an example, a force vector can be written as

$$\mathbf{F} = F^1\mathbf{a}_1 + F^2\mathbf{a}_2 \qquad \text{Contravariant Vector} \tag{4.33}$$

or

$$\mathbf{F} = F_1\mathbf{a}^1 + F_2\mathbf{a}^2 \qquad \text{Covariant Vector} \tag{4.34}$$

Notice that the contravariant vector is represented in terms of the contravariant components and the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, while the covariant vector is represented in terms of the covariant components and the base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$. Either base system or both may be used in connection with the vectorial treatment of a given problem.

As an example, the work done in moving an object through a displacement $\mathbf{r}$ by a force $\mathbf{F}$ can be expressed three ways:

(a) in terms of contravariant vectors
(b) in terms of covariant vectors
(c) in terms of a mixture of contravariant and covariant vectors.

(a) Work done using contravariant vectors.

The work done in terms of the contravariant vectors is

$$\mathrm{W} = \mathbf{F} \bullet \mathbf{r} = (F^1\mathbf{a}_1 + F^2\mathbf{a}_2) \bullet (x^1\mathbf{a}_1 + x^2\mathbf{a}_2) \tag{4.35}$$

$$\mathrm{W} = F^1x^1\, \mathbf{a}_1 \bullet \mathbf{a}_1 + F^1x^2\, \mathbf{a}_1 \bullet \mathbf{a}_2 + F^2x^1\, \mathbf{a}_2 \bullet \mathbf{a}_1 + F^2x^2\, \mathbf{a}_2 \bullet \mathbf{a}_2$$

Now

$$\mathbf{a}_1 \bullet \mathbf{a}_1 = |\mathbf{a}_1|\,|\mathbf{a}_1|\cos 0^0 = 1 \tag{4.36}$$

$$\mathbf{a}_2 \bullet \mathbf{a}_2 = |\mathbf{a}_2|\,|\mathbf{a}_2|\cos 0^0 = 1 \tag{4.37}$$

If the angle between the two axis is α then

$$\mathbf{a}_1 \bullet \mathbf{a}_2 = |\mathbf{a}_1|\,|\mathbf{a}_2|\cos\alpha = \cos\alpha \tag{4.38}$$

$$\mathbf{a}_2 \bullet \mathbf{a}_1 = |\mathbf{a}_2|\,|\mathbf{a}_1|\cos\alpha = \cos\alpha \tag{4.39}$$

Therefore the work done in terms of the contravariant components is

$$\mathrm{W} = F^1x^1 + F^2x^2 + (F^1x^2 + F^2x^1)\cos\alpha \tag{4.40}$$

which is not particularly simple and is dependent upon the angle α.

(b) The work done using covariant vectors.

The work done in terms of the covariant vectors is

$$\mathrm{W} = \mathbf{F} \bullet \mathbf{r} = (F_1\mathbf{a}^1 + F_2\mathbf{a}^2) \bullet (x_1\mathbf{a}^1 + x_2\mathbf{a}^2) \tag{4.41}$$

$$\mathrm{W} = F_1x_1\, \mathbf{a}^1 \bullet \mathbf{a}^1 + F_1x_2\, \mathbf{a}^1 \bullet \mathbf{a}^2 + F_2x_1\, \mathbf{a}^2 \bullet \mathbf{a}^1 + F_2x_2\, \mathbf{a}^2 \bullet \mathbf{a}^2 \tag{4.42}$$

Now we showed in equations 4.13 and 4.14 that

$$\mathbf{a}^1 = \frac{\mathbf{e}^1}{\sin\alpha} \quad \text{and} \quad \mathbf{a}^2 = \frac{\mathbf{e}^2}{\sin\alpha}$$

Therefore,

$$\mathbf{a}^1 \bullet \mathbf{a}^1 = \left|\frac{1}{\sin\alpha}\right|\left|\frac{1}{\sin\alpha}\right|\cos 0^0 = \frac{1}{\sin^2\alpha} \tag{4.43}$$

and

$$\mathbf{a}^2 \bullet \mathbf{a}^2 = \left|\frac{1}{\sin\alpha}\right|\left|\frac{1}{\sin\alpha}\right|\cos 0^0 = \frac{1}{\sin^2\alpha} \tag{4.44}$$

Now the angle between $\mathbf{a}^1$ and $\mathbf{a}^2$ is $(180^0 - \alpha)$ as can be seen in figure 4.7. Therefore

$$\mathbf{a}^1 \bullet \mathbf{a}^2 = \left|\frac{1}{\sin\alpha}\right|\left|\frac{1}{\sin\alpha}\right|\cos(180^0 - \alpha)$$

but

$$\cos(180^0 - \alpha) = \cos(180^0)\cos\alpha - \sin(180^0)\sin(-\alpha) = -\cos\alpha$$

Therefore

$$\mathbf{a}^1 \bullet \mathbf{a}^2 = -\frac{\cos\alpha}{\sin^2\alpha} \tag{4.45}$$

$$\mathbf{a}^2 \bullet \mathbf{a}^1 = -\frac{\cos\alpha}{\sin^2\alpha} \tag{4.46}$$

Substituting equations 4.43 through 4.45 into equation 4.42 gives for the work done

$$W = \frac{F_1 x_1}{\sin^2\alpha} + \frac{F_2 x_2}{\sin^2\alpha} - \frac{F_2 x_1 \cos\alpha}{\sin^2\alpha} - \frac{F_1 x_2 \cos\alpha}{\sin^2\alpha} \tag{4.47}$$

which is a rather complicated form for the work done

(c) The work done using a mixture of contravariant and covariant components.

The work done can be expressed as the product of the contravariant force vector and the covariant displacement vector. That is,

$$W = \mathbf{F} \bullet \mathbf{r} = (F^1\mathbf{a}_1 + F^2\mathbf{a}_2) \bullet (x_1\mathbf{a}^1 + x_2\mathbf{a}^2) \tag{4.48}$$

$$W = F^1x_1\, \mathbf{a}_1 \bullet \mathbf{a}^1 + F_1x_2\, \mathbf{a}_1 \bullet \mathbf{a}^2 + F^2x_1\, \mathbf{a}_2 \bullet \mathbf{a}^1 + F^2x_2\, \mathbf{a}_2 \bullet \mathbf{a}^2 \tag{4.49}$$

But as can be seen in figure 4.7, and shown in equation 4.30

$$\mathbf{a}_1 \bullet \mathbf{a}^2 = 0 \qquad \text{because } \mathbf{a}_1 \perp \mathbf{a}^2$$
$$\mathbf{a}_2 \bullet \mathbf{a}^1 = 0 \qquad \text{because } \mathbf{a}_2 \perp \mathbf{a}^1$$

and by equation 4.29

$$\mathbf{a}_1 \bullet \mathbf{a}^1 = \mathbf{a}_2 \bullet \mathbf{a}^2 = 1$$

Replacing these values into equation 4.49 gives

$$W = F^1 x_1 + F^2 x_2 \tag{4.50}$$

Equation 4.50 gives the work done expressed in terms of contravariant and covariant components.

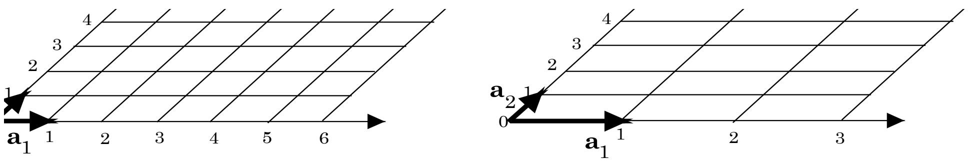
If we had expressed the force as a covariant vector and the displacement as the contravariant vector we would have obtained

$$W = \mathbf{F} \bullet \mathbf{r} = (F_1\mathbf{a}^1 + F_2\mathbf{a}^2) \bullet (x^1\mathbf{a}_1 + x^2\mathbf{a}_2)$$

$$W = F_1 x^1 + F_2 x^2 \tag{4.51}$$

In general the product of a contravariant vector with a covariant vector will yield an invariant (scalar) which will be independent of the coordinate system used. Hence, when using skewed coordinate systems, it is an advantage to have two reciprocal base systems. In most of the analysis done in general relativity by tensor analysis, there will usually be a mix of covariant and contravariant vectors.

Also note that the unitary vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ can have any magnitude. As an example if $|\mathbf{a}_2| = 1$ and $|\mathbf{a}_1| = 1$ then the space grid would look as in figure 4.33(a). If on the other hand $|\mathbf{a}_2| = 1$ and $|\mathbf{a}_1| = 2$, the space grid would appear as in figure 4.33(b). We see that this amounts to having a different scale on each axis. That is, a unit length on the x^1 axis is twice as large as the unit length on the x^2 axis. Although $\mathbf{a}_1$ and $\mathbf{a}_2$ can have any magnitude in general, we will almost always let them have unit magnitude.



(a) $|\mathbf{a}_1| = |\mathbf{a}_2| = 1$ Equal Spaced Grid

(b) $|\mathbf{a}_2| = 1$; $|\mathbf{a}_1| = 2$ Different scale on x^1 and x^2 axis.

Figure 4.3 The unitary vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ can have any magnitude.

Note that the contravariant vector in equation 4.33 has the contravariant components of the vector, F^1 and F^2, and the unitary base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$; while the covariant vector in equation 4.34 is represented in terms of the covariant components of the vector F_1 and F_2 and the reciprocal unitary base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$.

$$\mathbf{F} = F^1\mathbf{a}_1 + F^2\mathbf{a}_2 \qquad \text{Contravariant Vector} \tag{4.33}$$

and

$$\mathbf{F} = F_1\mathbf{a}^1 + F_2\mathbf{a}^2 \qquad \text{Covariant Vector} \tag{4.34}$$

Notice that the unitary base vectors are described with subscripts, while the reciprocal unitary base vectors are described with superscripts. Notice that the product of each term is a product of contravariant superscript and a covariant subscript. Hence, the vector can be thought of as consisting of contravariant and covariant terms; a superscript times a subscript for a contravariant vector and a subscript times a superscript for a covariant vector. Since the product of a contravariant vector and a covariant vector gives us an invariant quantity or a constant, this notation will give us an invariant quantity for the magnitude of any vector. We will see much more of this later.

In general, the product of a contravariant vector with a covariant vector will yield an invariant (scalar) which will be independent of the coordinate system used. Hence when using skewed coordinate systems, it is an advantage to have two reciprocal base systems. We will see that in most of the analysis done in general relativity, there will usually be a mix of covariant and contravariant vectors.

Summary of Basic Concepts

Skewed Coordinate Systems. In order to show inertial motion that is consistent with the Lorentz Transformation, it is necessary to draw coordinate systems that are skewed to each other rather than to use the traditional orthogonal coordinate systems.

Components of a vector in a rectangular coordinate system. The x-component of a vector is found by dropping a perpendicular line from the tip of **r** down to the x-axis. *Note that the line perpendicular to the x-axis is also parallel to the y-axis.* The y-component is found by dropping a perpendicular line from the tip of **r** to the y-axis. *Also note that the line perpendicular to the y-axis is parallel to the x-axis.*

Contravariant components of a vector in a skewed coordinate system. *For the skewed coordinate system the parallel of one axis is not perpendicular to the other. For the first component we drop a line from the tip of* **r**, ***parallel to the*** x^2 ***axis,*** to the x^1 axis. *This component is called the* ***contravariant*** *component of the vector* ***r*** *and is designated as* x^1. Drop another line, *parallel to the* x^1 *axis*, to the x^2 axis. This gives the second contravariant component x^2. The base vector $\mathbf{a}_1$ is in the direction of the x^1 axis and $\mathbf{a}_2$ is in the direction of the x^2 axis. The base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are called unitary vectors although they don't necessarily have to be unit vectors. **The vector r can be written in terms of the contravariant components as**

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2$$

Covariant components of a vector in a skewed coordinate system. For a second representation of the components of a vector in a skewed coordinate system, we *drop a line from the tip of* ***r, perpendicular*** *to the* x^1 *axis intersecting it at the point that we will now designate as* x_1, *We will call* x_1 *a covariant component of the vector* ***r***. We now *drop a line from the tip of* ***r perpendicular*** *to the* x^2 *axis, obtaining the second covariant component* x_2. We now have the two vector components $x_1\mathbf{a}_1$ and $x_2\mathbf{a}_2$. However, these vector components do not satisfy the parallelogram law of vector addition when we try to add them together head to tail. That is,

$$\mathbf{r} \neq x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2,$$

However, if we define two new base vectors

$$a^1 = \frac{e^1}{\sin\alpha} \quad \text{and} \quad a^2 = \frac{e^2}{\sin\alpha}$$

then the **vector r can be written in terms of the covariant components as**

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2$$

In general the product of a contravariant vector with a covariant vector will yield an invariant (scalar) which will be independent of the coordinate system used. Hence, when using skewed coordinate systems, it is an advantage to have two reciprocal base systems. We will see that in most of the analysis done in general relativity, there will usually be a mix of covariant and contravariant vectors.

In Summary, in a skewed coordinate system there are two types of components- contravariant and covariant. The contravariant components are found by parallel projections onto the coordinate axes while the covariant components are found by perpendicular projections. Contravariant components are designated by superscripts, x^i***, while covariant components are designated by subscripts,*** x_i**.** The base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$ are not unit vectors even if $\mathbf{a}_1$ and $\mathbf{a}_2$ are. The distinction between contravariant and covariant components disappears in orthogonal coordinates, because the axes are orthogonal. That is, in orthogonal coordinates, a projection which is parallel to one axis, is also perpendicular to the other.

Summary of Important Equations

The vector **r** written in terms of contravariant components $\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2$ (4.2)

Contravariant component x^1

$$x^1 = \frac{r \sin(\alpha - \theta)}{\sin\alpha} \quad (4.4)$$

Contravariant component x^2

$$x^2 = \frac{r \sin\theta}{\sin\alpha} \quad (4.6)$$

The base vectors

$$a^1 = \frac{e^1}{\sin\alpha} \quad (4.13)$$

$$a^2 = \frac{e^2}{\sin\alpha} \quad (4.14)$$

The vector **r** written in terms of covariant components $\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2$ (4.15)

Covariant component x_1

$$x_1 = r\cos\theta \quad (4.16)$$

Covariant component x_2

$$x_2 = r\cos(\alpha - \theta) \quad (4.17)$$

Lorentz transformation for space coordinates

$$x'' = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad (4.31)$$

Lorentz transformation for the time coordinates.

$$t'' = \frac{t' + x'\frac{v}{c^2}}{\sqrt{1 - v^2/c^2}} \quad (4.35)$$

Inverse Lorentz Transformation for space coordinates

$$x' = \frac{x'' - vt''}{\sqrt{1 - v^2/c^2}} \quad (4.36)$$

Inverse Lorentz Transformation for time coordinates

$$t' = \frac{t'' - x''\frac{v}{c^2}}{\sqrt{1 - v^2/c^2}} \quad (4.37)$$

Length contraction formula

$$L = L_0\sqrt{1 - v^2/c^2} \quad (4.46)$$

Time dilation formula

$$\Delta t'' = \frac{\Delta t_0'}{\sqrt{1 - v^2/c^2}} \quad (4.52)$$

$$\Delta t' = \frac{\Delta t_0''}{\sqrt{1 - v^2/c^2}} \quad (4.56)$$

The invariant interval of Spacetime

$$(\Delta s^2) = (\Delta x')^2 - c^2(\Delta t')^2 \quad (4.94)$$

and

$$(\Delta s^2) = (\Delta x'')^2 - c^2(\Delta t'')^2 \quad (4.95)$$

The invariant interval of Spacetime in terms of differential quantities

$$(ds^2) = (dx')^2 - c^2(dt')^2 \quad (4.96)$$

and

$$(ds^2) = (dx'')^2 - c^2(dt'')^2 \quad (4.97)$$

The invariant interval in four-dimensional spacetime

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \quad (4.98)$$

Reciprocal unitary vectors expressed in terms of the unitary vectors

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \quad (4.16)$$

$$\mathbf{a}^2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \quad (4.17)$$

$$\mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \quad (4.18)$$

Products of the unitary vectors and the reciprocal unitary vectors

$$\mathbf{a}^1 \bullet \mathbf{a}_1 = \mathbf{a}^2 \bullet \mathbf{a}_2 = \mathbf{a}^3 \bullet \mathbf{a}_3 = 1 \quad (4.24)$$

$$\mathbf{a}^1 \bullet \mathbf{a}_2 = \mathbf{a}^1 \bullet \mathbf{a}_3 = \mathbf{a}^2 \bullet \mathbf{a}_1 = \mathbf{a}^2 \bullet \mathbf{a}_3 = \mathbf{a}^3 \bullet \mathbf{a}_1 = \mathbf{a}^3 \bullet \mathbf{a}_2 = 0 \quad (4.25)$$

The unitary vectors expressed in terms of the reciprocal unitary vectors,

$$\mathbf{a}_1 = \frac{\mathbf{a}^2 \times \mathbf{a}^3}{\mathbf{a}^1 \bullet (\mathbf{a}^2 \times \mathbf{a}^3)} \quad (4.26)$$

$$\mathbf{a}_2 = \frac{\mathbf{a}^3 \times \mathbf{a}^1}{\mathbf{a}^1 \bullet (\mathbf{a}^2 \times \mathbf{a}^3)} \quad (4.27)$$

$$\mathbf{a}_3 = \frac{\mathbf{a}^1 \times \mathbf{a}^2}{\mathbf{a}^1 \bullet \left(\mathbf{a}^2 \times \mathbf{a}^3\right)} \qquad (4.28)$$

The combinations of all the products

$$\mathbf{a}^1 \bullet \mathbf{a}_1 = \mathbf{a}_1 \bullet \mathbf{a}^1 = \mathbf{a}^2 \bullet \mathbf{a}_2 = \mathbf{a}_2 \bullet \mathbf{a}^2 = \mathbf{a}^3 \bullet \mathbf{a}_3 = \mathbf{a}_3 \bullet \mathbf{a}^3 = 1 \qquad (4.29)$$

and

$$\mathbf{a}^1 \bullet \mathbf{a}_2 = \mathbf{a}_1 \bullet \mathbf{a}^2 = \mathbf{a}^1 \bullet \mathbf{a}_3 = \mathbf{a}_1 \bullet \mathbf{a}^3 = \mathbf{a}^2 \bullet \mathbf{a}_1 = \mathbf{a}_2 \bullet \mathbf{a}^1 = \mathbf{a}^2 \bullet \mathbf{a}_3 = \mathbf{a}_2 \bullet \mathbf{a}^3 = \mathbf{a}^3 \bullet \mathbf{a}_1 = \mathbf{a}_3 \bullet \mathbf{a}^1 = \mathbf{a}^3 \bullet \mathbf{a}_2 = \mathbf{a}_3 \bullet \mathbf{a}^2 = 0 \qquad (4.29)$$

For a three dimensional skewed coordinate system, the vector **r** is written *in terms of the three dimensional* ***contravariant components*** *x^1, x^2, and x^3, and the unitary vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ as*

$$\mathbf{r} = x^1\mathbf{a}_1 + x^2\mathbf{a}_2 + x^3\mathbf{a}_3 \qquad (4.31)$$

*The vector **r** can also be expressed in terms of the* ***covariant components*** *x_1, x_2, and x_3 of the vector and the reciprocal system of unitary vectors $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ as*

$$\mathbf{r} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 + x_3\mathbf{a}^3 \qquad (4.32)$$

Work done using contravariant vectors.

$$W = \mathbf{F} \bullet \mathbf{r} = (F^1\mathbf{a}_1 + F^2\mathbf{a}_2) \bullet (x^1\mathbf{a}_1 + x^2\mathbf{a}_2) \qquad (4.35)$$

$$W = F^1x^1 + F^2x^2 + (F^1x^2 + F^2x^1)\cos\alpha \qquad (4.40)$$

Work done using covariant vectors.

$$W = \mathbf{F} \bullet \mathbf{r} = (F_1\mathbf{a}^1 + F_2\mathbf{a}^2) \bullet (x_1\mathbf{a}^1 + x_2\mathbf{a}^2) \qquad (4.41)$$

$$W = \frac{F_1x_1}{\sin^2\alpha} + \frac{F_2x_2}{\sin^2\alpha} - \frac{F_2x_1\cos\alpha}{\sin^2\alpha} - \frac{F_1x_2\cos\alpha}{\sin^2\alpha} \qquad (4.47)$$

The work done using a mixture of contravariant and covariant components.

$$W = \mathbf{F} \bullet \mathbf{r} = (F^1\mathbf{a}_1 + F^2\mathbf{a}_2) \bullet (x_1\mathbf{a}^1 + x_2\mathbf{a}^2) \qquad (4.48)$$

$$W = F^1 x_1 + F^2 x_2 \qquad (4.50)$$

$$W = F_1 x^1 + F_2 x^2 \qquad (4.51)$$

Questions for Chapter 4

1. Why can't we just use orthogonal systems in our analysis of relativity?
2. What is a contravariant vector?
3. What is a covariant vector?
4. Is a unitary vector the same as a unit vector?
5. When using a product of two vectors, is it better to have two covariant vectors, two contravariant vectors, or one of each?

Problems for Chapter 4

4.1 The Components of a Vector in Skewed Coordinates

1. A vector **r** has a magnitude of 25.0 units and makes an angle of 55.0^0 with the x-axis. Find the rectangular components of this vector

2. A vector **r** has a magnitude of 25.0 units and makes an angle of 55.0^0 with the x-axis. If the skewed coordinate system, makes an angle $\alpha = 35.0^0$, (a) find the contravariant components of this vector, and (b) express the vector in terms of its contravariant components.

3. A vector **r** has a magnitude of 25.0 units and makes an angle of 55.0^0 with the x-axis. If the skewed coordinate system makes an angle $\alpha = 35.0^0$, (a) find the covariant components of this vector, (b) express the vector in terms of its covariant components, and (c) find the values of the base vectors.

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