Chapter 2 Kinematics In One Dimension

My purpose is to set forth a very new science dealing with a very ancient subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by Philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated ... and what I consider more important, there has been opened up to this vast and most excellent science, of which my work is merely the beginning, ways and means by which other minds more acute than mine will explore its remote corners.

Galileo Galilei
Dialogues Concerning Two New Sciences

2.1 Kinematics - The Study of Motion

Kinematics is defined as that branch of mechanics that studies the motion of a body without regard to the cause of that motion. In our everyday life we constantly observe objects in motion. For example, an object falls from the table, a car moves along the highway, or a plane flies through the air. In this process of motion, we observe that at one time the object is located at one position in space and then at a later time it has been displaced to some new position. Motion thus entails a movement from one position to another position. Therefore, to describe the motion of a body logically, we need to start by defining the position of a body. To do this we need a reference system. Thus, we introduce a coordinate system, as shown in figure 2.2. The body is located at the point 0 at the time \( t = 0 \). The point 0, the origin of the coordinate system, is the reference position. We measure the displacement of the moving body from there. After an elapse of time \( t_1 \) the object will have moved from 0 and will be found along the x-axis at position 1, a distance \( x_1 \) away from 0.

A little later in time, at \( t = t_2 \), the object will be located at point 2, a distance \( x_2 \) away from 0. (As an example, the moving body might be a car on the street. The
Figure 2.2 The position of an object at two different times.

The average velocity of the body in motion between the points 1 and 2 is defined as the displacement of the moving body divided by the time it takes for that displacement. That is,

$$v_{\text{avg}} = \frac{\text{displacement}}{\text{time for displacement}}$$  \hspace{1cm} (2.1)

where $v_{\text{avg}}$ is the notation used for the average velocity. The displacement is the distance that the body moves in a specified direction. For this description of one-dimensional motion, the displacement is positive if the distance the body has moved is in the positive $x$-direction. Conversely, the displacement is negative if the distance the body has moved is in the negative $x$-direction. Hence a positive value of $v$ implies a velocity in the positive $x$-direction, while a negative value of $v$ implies a velocity in the negative $x$-direction. If the one-dimensional motion is in the $y$-direction, the displacement is positive if the distance the body has moved is in the positive $y$-direction. Conversely, the displacement is negative if the distance the body has moved is in the negative $y$-direction. Also a positive value of $v$ implies a velocity in the positive $y$-direction, while a negative value of $v$ implies a velocity in the negative $y$-direction. The more general case, the velocity of a moving body in two dimensions, is treated in chapter 4.

From figure 2.2, we can see that during the time interval $t_2 - t_1$, the displacement or change in position of the body is simply $x_2 - x_1$. Therefore, the average velocity of the body in motion between points 1 and 2 is

$$v_{\text{avg}} = \frac{x_2 - x_1}{t_2 - t_1}$$  \hspace{1cm} (2.2)

Note here that in the example of the car and the telephone poles, $t_1$ is the time on a clock when the car passes the first telephone pole, position 1, and $t_2$ is the time on the same clock when the car passes the second telephone pole, position 2.
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A convenient notation to describe this change in position with the change in time is the delta notation. Delta (the Greek letter \( \Delta \)) is used as a symbolic way of writing “change in,” that is,

\[
\Delta x = \text{(change in } x) = x_2 - x_1 \tag{2.3}
\]

and

\[
\Delta t = \text{(change in } t) = t_2 - t_1 \tag{2.4}
\]

Using this delta notation we can write the average velocity as

\[
v_{\text{avg}} = \frac{\Delta x}{\Delta t} \tag{2.5}
\]

**Example 2.1**

Finding the average velocity using the \( \Delta \) notation. A car passes telephone pole number 1, located 20.0 m down the street from the corner lamp post, at a time \( t_1 = 8.00 \text{ s} \). It then passes telephone pole number 2, located 80.0 m from the lamp post, at a time of \( t_2 = 16.0 \text{ s} \). What was the average velocity of the car between the positions 1 and 2?

**Solution**

The average velocity of the car, found from equation 2.5, is

\[
v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{80.0 \text{ m} - 20.0 \text{ m}}{16.0 \text{ s} - 8.00 \text{ s}} = \frac{60.0 \text{ m}}{8.00 \text{ s}} = 7.5 \text{ m/s}
\]

(Note that according to the convention that we have adopted, the 7.5 m/s represents a velocity in the positive \( x \)-direction. If the answer were \(-7.5 \text{ m/s}\) the direction of the velocity would have been in the negative \( x \)-direction.)

To go to this interactive example click on this sentence.

For convenience, the reference position 0 that is used to describe the motion is occasionally moved to position 1, then \( x_1 = 0 \), and the displacement is denoted by \( x \), as shown in figure 2.3. The clock is started at this new reference position 1, so \( t_1 = 0 \) there. We now express the elapsed time for the displacement as \( t \). In this simplified coordinate system the average velocity is
Figure 2.3 The position of an object determined from a new reference system.

\[ v_{avg} = \frac{x}{t} \]  

Remember, the average velocity is the same physically in both equations 2.5 and 2.6; the numerator is still the displacement of the moving body, and the denominator is still the elapsed time for this displacement. Because the reference point has been changed, the notation appears differently. We use both notations in the description of motion. The particular notation we use depends on the problem.

Example 2.2

Changing the reference position. A car passes telephone pole number 1 at \( t = 0 \) on a watch. It passes a second telephone pole 60.0 m down the block 8.00 seconds later. What is the car’s average velocity?

Solution

The average velocity, found from equation 2.6, is

\[ v_{avg} = \frac{x}{t} = \frac{60.0 \text{ m}}{8.00 \text{ s}} = 7.5 \text{ m/s} \]

Also note that this is the same problem solved in accept1; only the reference position for the measurement of the motion has been changed.

To go to this interactive example click on this sentence.

Before we leave this section, we should make a distinction between the average velocity of a body and the average speed of a body. The average speed of a
body is the distance that a body moves per unit time. The average velocity of a body is the displacement of a body per unit time. Because the displacement of a body specifies the distance an object moves in a specified direction, its velocity is also in that direction. The speed is just the distance traveled divided by the time and does not specify a direction for the motion. For example, if a girl runs 100 m in the \( x \)-direction and turns around and returns to the starting point in a total time of 90 s, her average velocity is zero because her displacement is zero. Her average speed, on the other hand, is the total distance she ran divided by the total time it took, or 200 m/90 s = 2.2 m/s. If she ran 100 m in 45 s in one direction only, let us say the positive \( x \)-direction, her average speed is 100 m/45 s = 2.2 m/s. Her average velocity is 2.2 m/s in the positive \( x \)-direction. Speed is always a positive quantity, whereas velocity can be either positive or negative depending on whether the motion is in the positive \( x \)-direction or the negative \( x \)-direction, respectively.

Section 2.2 shows how the motion of a body can be studied in more detail in the laboratory.

### 2.2 Experimental Description of a Moving Body

Following Galileo’s advice that motion should be studied by experiment, let us go into the laboratory and describe the motion of a moving body on an air track\(^1\). An air track is a hollow aluminum track. Air is forced into the air track by a blower and flows out the sides of the track through many small holes. When a glider is placed on the track, the air escaping from the holes in the track provides a cushion of air for the glider to move on, thereby substantially reducing the retarding force of friction on the glider. The setup of an air track in the laboratory is shown in figure 2.4.

![Setup of an airtrack.](image)

**Figure 2.4** Setup of an airtrack.

We connect a spark timer, a device that emits electrical pulses at certain prescribed times, to a wire on the air track. A piece of spark-timer tape is attached to the air track to act as a permanent record of the position of the moving glider as a function of time. A spark from the timer jumps across an air gap between the

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\(^1\)For a more detailed description of such an experiment in kinematics on an air track see, “Experiments in Physics” 2ed by Nolan and Bigliani, Whittier Publishers.
glider wire and the air track, and in so doing it burns a hole in the timer tape. This burned hole on the tape, which appears as a dot, is a record of the position of the glider at that instant of time. Thus, the combination of a glider, an air track, and a spark timer gives us a record of the position of a moving body at any instant of time. Let us now look at an experiment with a glider moving at constant velocity along the air track.

2.3 A Body Moving at Constant Velocity
To study a body moving at constant velocity we place a glider on a level air track and give it a slight push to initiate its motion along the track. The spark timer is turned on, leaving a permanent record of this motion on a piece of spark-timer tape. The distance traveled by the glider as a function of time is recorded on the spark-timer paper, and appears as in figure 2.5. The spark timer is set to give a spark every 1/30 of a second. The first dot occurs at the time \( t = 0 \), and each succeeding dot occurs at a time interval of 1/30 of a second later. We label the first dot as dot 0, the reference position, and then measure the total distance \( x \) from the first dot to each succeeding dot with a meter stick.

![Figure 2.5 Spark-timer paper showing constant velocity.](image)

The measured data for the total distance traveled by the glider as a function of time are plotted in figure 2.6. Note that the plot is a straight line. If you measure the slope of this line you will observe that it is \( \Delta x/\Delta t \), which is the average velocity defined in equation 2.5. Since all the points generate a straight line, which has a constant slope, the velocity of the glider is a constant equal to the slope of this graph. Whenever a body moves in such a way that it always travels equal distances in equal times, that body is said to be moving with a constant velocity. This can also be observed in figure 2.5 by noting that the dots are equally spaced. The SI unit for velocity is m/s. The units cm/s and km/hr are also used. Note that on a graph of the displacement \( x \) of a moving body versus time \( t \), the slope \( \Delta x/\Delta t \) always represents a velocity. If the slope is positive, the velocity is positive and the direction of the moving body is toward the right. If the slope is negative, the velocity is negative and the direction of the moving body is toward the left.
Example 2.3

The velocity of a glider on an air track. A glider goes from a position of 20.4 cm at a time of \( t = 10/30 \) s to a position of 103 cm at a time of \( t = 50/30 \) s. Find the average velocity of the glider during this interval.

Solution

The average velocity of the glider, found from equation 2.5, is

\[
\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}
\]

\[
= \frac{103 \text{ cm} - 20.4 \text{ cm}}{50/30 \text{ s} - 10/30 \text{ s}} = \frac{82.6 \text{ cm}}{4/3 \text{ s}}
\]

\[
= 62.0 \text{ cm/s}
\]

To go to this interactive example click on this sentence.

2.4 A Body Moving at Constant Acceleration

If we tilt the air track at one end it effectively becomes a frictionless inclined plane. We place a glider at the top of the track and then release it from rest. Figure 2.7 is a picture of the glider in its motion on the inclined air track.
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Figure 2.7 The tilted air track.

The spark timer is turned on, giving a record of the position of the moving glider as a function of time, as illustrated in figure 2.8. The most important feature to immediately note on this record of the motion, is that the dots, representing the positions of the glider, are no longer equally spaced as they were for motion at constant speed, but rather become farther and farther apart as the time increases. The total distance \( x \) that the glider moves is again measured as a function of time. If we plot this measured distance \( x \) against the time \( t \), we obtain the graph shown in figure 2.9.

The first thing to note in this figure is that the graph of \( x \) versus \( t \) is not a straight line. However, as you may recall from section 2.3, the slope of the distance versus time graph, \( \Delta x/\Delta t \), represents the velocity of the moving body. But in figure 2.9 there are many different slopes to this curve because it is continuously changing with time. Since the slope at any point represents the velocity at that point, we observe that the velocity of the moving body is changing with time. The change of velocity with time is defined as the acceleration of the moving body, and the average acceleration is written as

\[
g_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{2.7}
\]
Because the velocity is changing continuously, the average velocity for every time interval can be computed from equation 2.5. Thus, subtracting each value of \( x \) from the next value of \( x \) gives us \( \Delta x \), the distance the glider moves during one time interval. The average velocity during that interval can then be computed from \( v_{\text{avg}} = \frac{\Delta x}{\Delta t} \). At the beginning of this interval the actual velocity is less than this value while at the end of the interval it is greater. Later we will see that for constant acceleration, the velocity at the center of the time interval is equal to the average velocity for the entire time interval.

If we plot the velocity at the center of the interval against the time, we obtain the graph in figure 2.10. We can immediately observe that the graph is a straight line. The slope of this line, \( \Delta v/\Delta t \), is the experimental acceleration of the glider. Since this graph is a straight line, the slope is a constant; this implies that the acceleration is also a constant. Hence, the acceleration of a body moving down a frictionless inclined plane is a constant. Because in constantly accelerated motion the average acceleration is the same as the constant acceleration, the subscript \( \text{avg} \)
will be deleted from the acceleration \( a \) in equation 2.7 and in all the equations dealing with this type of motion.

Since acceleration is a change in velocity per unit time, the units for acceleration are velocity divided by the time. In SI units, the acceleration is

\[
\frac{\text{m/s}}{\text{s}}
\]

For convenience, this is usually written in the equivalent algebraic form as \( \text{m/s}^2 \). But we must not forget the physical meaning of a change in velocity of so many \( \text{m/s} \) every second. Other units used to express acceleration are \( \text{cm/s}^2 \), and \( \text{(km/hr)/s} \).

Example 2.4

The acceleration of a glider on an air track. A glider’s velocity on a tilted air track increases from 3.83 cm/s at the time \( t = 10/30 \) s to 42.3 cm/s at a time of \( t = 70/30 \) s. What is the acceleration of the glider?

Solution

The acceleration of the glider, found from equation 2.7, is

\[
a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{42.3 \text{ cm/s} - 3.83 \text{ cm/s}}{70/30 \text{ s} - 10/30 \text{ s}} = \frac{38.5 \text{ cm/s}}{6/3 \text{ s}} = 19.2 \text{ cm/s}^2
\]

Before leaving this section we should note that if the acceleration is a positive quantity, the velocity is increasing with time. If the acceleration is a negative quantity, the velocity is decreasing with time. When the velocity is positive, indicating that the body is moving in the positive \( x \)-direction, and the acceleration is positive, the object is speeding up, or accelerating. However, when the velocity is positive, and the acceleration is negative, the object is slowing down, or decelerating. On the other hand, if the velocity is negative, indicating that the body is moving in the negative \( x \)-direction, and the acceleration is negative, the body is speeding up in the negative \( x \)-direction. However, when the velocity is negative and the acceleration is positive, the body is slowing down in the negative \( x \)-direction. If the acceleration lasts long enough, the body will eventually come to a stop and will then start moving in the positive \( x \)-direction. The velocity will then be positive and the body will be speeding up in the positive \( x \)-direction.
2.5 The Instantaneous Velocity and Instantaneous Acceleration of a Moving Body

In section 2.4 we observed that the velocity of the glider varies continuously as it “slides” down the frictionless inclined plane. We also stated that the average velocity could be computed from $v_{avg} = \Delta x/\Delta t$. At the beginning of the interval of motion the actual velocity is less than this value while at the end of the interval it is greater. If the interval is made smaller and smaller, the average velocity $v_{avg}$ throughout the interval becomes closer to the actual velocity at the instant the body is at the center of the time interval. Finding the velocity at a particular instant of time leads us to the concept of instantaneous velocity. **Instantaneous velocity** is defined as the limit of $\Delta x/\Delta t$ as $\Delta t$ gets smaller and smaller, eventually approaching zero. We write this concept mathematically as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad (2.8)$$

*Note that this is the same as the definition of the derivative of $x$ with respect to $t$ defined in your calculus course*, a concept that we will use throughout the book. As in the case of average velocity in one-dimensional motion, if the limit of $\Delta x/\Delta t$ is a positive quantity, the velocity is toward the right. If the limit of $\Delta x/\Delta t$ is a negative quantity, the velocity is toward the left.

The concept of instantaneous velocity can be easily understood by performing the following experiment on an air track². First, we tilt the air track to again give an effectively frictionless inclined plane. Then we place a 20-cm length of metal, called a flag, at the top of the glider. A photocell gate, which is a device that can be used to automatically turn a clock on and off, is attached to a clock timer and is placed on the air track. We then allow the glider to slide down the track. When the flag of the glider interrupts the light beam to the photocell, the clock is turned on. When the flag has completely passed through the light beam, the photocell gate turns off the clock. The clock thus records the time for the 20-cm flag to pass through the photocell gate. We find the average velocity of the flag as it moves through the gate from equation 2.5 as $v = \Delta x/\Delta t$. The 20-cm length of the flag is $\Delta x$, and $\Delta t$ is the time interval, as read from the clock.

We repeat the process for a 15-cm, 10-cm, and a 5-cm flag. For each case we measure the time $\Delta t$ that it takes for the flag to move through the gate. The first thing that we observe is that the time for the flag to move through the gate, $\Delta t$, gets smaller for each smaller flag. You might first expect that if $\Delta t$ approaches 0, the ratio of $\Delta x/\Delta t$ should approach infinity. However, since $\Delta x$, the length of the flag, is also getting smaller, the ratio of $\Delta x/\Delta t$ remains finite. If we plot $\Delta x/\Delta t$ as a function of $\Delta t$ for each flag, we obtain the graph in figure 2.11.

²See Experiment 7 in “Experiments in Physics” by Nolan and Bigliani, Whittier Publishers.
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Figure 2.11 Graph of $\Delta x/\Delta t$ versus $\Delta t$ to obtain the instantaneous velocity of the glider.

Notice that as $\Delta t$ approaches 0, $(\Delta t \to 0)$, the plotted line intersects the $\Delta x/\Delta t$ axis. At this point, the distance interval $\Delta x$ has been reduced from 20 cm to effectively 0 cm. The value of $\Delta t$ has become progressively smaller so this point represents the limiting value of $\Delta x/\Delta t$ as $\Delta t$ approaches 0. But this limit is the definition of the instantaneous velocity. Hence, the point where the line intersects the $\Delta x/\Delta t$ axis gives the value of the velocity of the glider at the instant of time that the glider is located at the position of the photocell gate. This limiting process allows us to describe the motion of a moving body in terms of the velocity of the body at any instant of time rather than in terms of the body’s average velocity.

Usually we will be more interested in the instantaneous velocity of a moving body than its average velocity. The speedometer of a moving car is a physical example of instantaneous velocity. Whether the car’s velocity is constant or changing with time, the instant that the speedometer is observed, the speedometer indicates the speed of the car at that particular instant of time. The instantaneous velocity of the car is that observed value of the speed in the direction that the car is traveling.

In a similar vein, a body can also have its acceleration changing with time. In that case the instantaneous acceleration is defined as the limit of $\Delta v/\Delta t$ as $\Delta t$ gets smaller and smaller, eventually approaching zero. We write this concept mathematically as

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad (2.9)$$

Note that this is the same as the definition of the derivative of $v$ with respect to $t$ defined in your calculus course. We will use this form for the acceleration latter in this chapter.

2.6 The Kinematic Equations
Because the previous experiments were based on motion at constant acceleration, we can only apply the results of those experiments to motion at a constant acceleration. Let us now compile those results into a set of equations, called the **kinematic equations of linear motion** that will describe the motion of a moving body. For motion at constant acceleration, the average acceleration is equal to the constant acceleration. Hence, the subscript avg can be deleted from equation 2.7 and that equation now gives the constant acceleration of the moving body as

\[
a = \frac{v_2 - v_1}{t_2 - t_1}
\]  

Equation 2.7 indicates that at the time \(t_1\) the body is moving at the velocity \(v_1\), while at the time \(t_2\) the body is moving at the velocity \(v_2\). This motion is represented in figure 2.12(a) for a runner.

Let us change the reference system by starting the clock at the time \(t_1 = 0\), as shown in figure 2.12(b). We will now designate the velocity of the moving body at the time 0 as \(v_0\) instead of the \(v_1\) in the previous reference system of figure 2.12(a). Similarly, the time \(t_2\) will correspond to any time \(t\) and the velocity \(v_2\) will be denoted by \(v\), the velocity at that time \(t\). Thus, the velocity of the moving body will be \(v_0\) when the time is equal to 0, and \(v\) when the time is equal to \(t\). This change of reference system allows us to rewrite equation 2.7 as

\[
a = \frac{v - v_0}{t}
\]  

**Figure 2.12** Change in reference system.
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Equation 2.10 is similar to equation 2.7 in that it gives the same definition for acceleration, namely a change in velocity with time, but in a slightly different but equivalent notation. Solving equation 2.10 for \( v \) gives the first of the very important kinematic equations, namely,

\[
v = v_0 + at
\]  
(2.11)

Equation 2.11 says that the velocity \( v \) of the moving object can be found at any instant of time \( t \) once the acceleration \( a \) and the initial velocity \( v_0 \) of the moving body are known.

Example 2.5

Using the kinematic equation for the velocity as a function of time. A car passes a green traffic light while moving at a velocity of 6.00 m/s. It then accelerates at 0.300 m/s\(^2\) for 15.0 s. What is the car’s velocity at 15.0 s?

Solution

The velocity, found from equation 2.11, is

\[
v = v_0 + at
\]

\[
= \left( 6.00 \text{ m/s} \right) + \left( 0.300 \text{ m/s}^2 \right)(15.0 \text{ s})
\]

\[
= 10.5 \text{ m/s}
\]

The velocity of the car is 10.5 m/s. This means that the car is moving at a speed of 10.5 m/s in the positive \( x \)-direction.

To go to this interactive example click on this sentence.

In addition to the velocity of the moving body at any time \( t \), we would also like to know the location of the body at that same time. That is, let us obtain an equation for the displacement of the moving body as a function of time. Solving equation 2.6 for the displacement \( x \) gives

\[
x = v_{avg} t
\]

Hence, the displacement of the moving body is equal to the average velocity of the body times the time it is in motion. For example, if you are driving your car at an average velocity of 50 km/hr, and you drive for a period of time of two hours, then your displacement is
You have traveled a total distance of 100 km from where you started.

Equation 2.12 gives us the displacement of the moving body in terms of its average velocity. The actual velocity during the motion might be greater than or less than the average value. The average velocity does not tell us anything about the body’s acceleration. We would like to express the displacement of the body in terms of its acceleration during a particular time interval, and in terms of its initial velocity at the beginning of that time interval.

For example, consider a car in motion along a road between the times \( t = 0 \) and \( t = t \). At the beginning of the time interval the car has an initial velocity \( v_0 \), while at the end of the time interval it has the velocity \( v \), as shown in figure 2.13. If the acceleration of the moving body is constant, then the average velocity throughout the entire time interval is

\[
\bar{v} = \frac{v_0 + v}{2}
\]

This averaging of velocities for bodies moving at constant acceleration is similar to determining a grade in a course. For example, if you have two test grades in the course, your course grade, the average of the two test grades, is the sum of the test grades divided by 2,

\[
\text{Avg. Grade} = \frac{100 + 90}{2} = 95
\]

If we substitute this value of the average velocity into equation 2.12, the displacement becomes

\[
x = \bar{v} t = \left( \frac{v_0 + v}{2} \right) t
\]
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Note that $v$ represents the final value of the velocity at the time $t$, the end of the time interval. But there already exists an equation for the value of $v$ at the time $t$, namely equation 2.11. Therefore, substituting equation 2.11 into equation 2.13 gives

$$x = \left[ \frac{v_0 + (v_0 + at)}{2} \right] t$$

Simplifying, we get

$$x = \left( \frac{2v_0 + at}{2} \right) t = \frac{2v_0t + \frac{1}{2}at^2}{2} = v_0t + \frac{1}{2}at^2$$

Equation 2.14, the second of the kinematic equations, represents the displacement $x$ of the moving body at any instant of time $t$. In other words, if the original velocity and the constant acceleration of the moving object are known, then we can determine the location of the moving object at any time $t$. Notice that the first term represents the distance that the moving body would travel if there were no acceleration and the body just moved at the constant velocity $v_0$ for the time $t$. The second term shows how far the body moves because there is an acceleration. If there were no initial velocity, that is $v_0 = 0$, this is the distance that the body will move because of the acceleration. In general, however, there is both an initial velocity and an acceleration, and the total displacement $x$ is the total distance that the body moves because of the two effects. This rather simple equation contains a tremendous amount of information.

**Example 2.6**

Using the kinematic equation for the displacement as a function of time. A car, initially traveling at 30.0 km/hr, accelerates at the constant rate of 1.50 m/s$^2$. How far will the car travel in 15.0 s?

**Solution**

To express the result in the proper units, km/hr is converted to m/s as

$$v_0 = 30.0 \frac{\text{km}}{\text{hr}} \left( \frac{1 \text{ hr}}{3600 \text{ s}} \right) \left( \frac{1000 \text{ m}}{1 \text{ km}} \right) = 8.33 \text{ m/s}$$

The displacement of the car, found from equation 2.14, is

$$x = v_0t + \frac{1}{2}at^2$$
The first term in the answer, 125 m, represents the distance that the car would travel if there were no acceleration and the car continued to move at the velocity 8.33 m/s for 15.0 s. But there is an acceleration, and the second term shows how much farther the car moves because of that acceleration, namely 169 m. The total displacement of 294 m is the total distance that the car travels because of the two effects.

**To go to this interactive example click on this sentence.**

As a further example of the kinematics of a moving body, consider the car moving along a road at an initial velocity of 95 km/hr = 26.4 m/s, as shown in figure 2.14. The driver sees a tree fall into the middle of the road 60.0 m away. The driver

**Figure 2.14** A tree falls on the road.
immediately steps on the brakes, and the car starts to decelerate at the constant rate of \( a = -5.50 \, \text{m/s}^2 \). (As mentioned previously, in one-dimensional motion a negative acceleration means that the acceleration is toward the left, in the opposite direction of the motion. If the velocity is positive, a negative value for the acceleration means that the body is slowing down or decelerating.) Will the car come to a stop before hitting the tree?

What we need for the solution of this problem is the actual distance the car travels before it can come to a stop while decelerating at the rate of 5.50 m/s\(^2\). Before we can find that distance, however, we must know the time it takes for the car to come to a stop. Then we substitute this stopping time into equation 2.14, and the equation tells us how far the car will travel before coming to a stop. (Note that most of the questions that might be asked about the motion of the car can be answered using the kinematic equations 2.11 and 2.14.)

Equation 2.11 tells us the velocity of the car at any instant of time. But when the car comes to rest its velocity is zero. Thus, at the time when the car comes to a stop \( t_{\text{stop}} \), the velocity \( v \) will be equal to zero. Therefore, equation 2.11 becomes

\[
0 = v_0 + at_{\text{stop}}
\]

Solving for the time for the car to come to a stop, we have

\[
t_{\text{stop}} = \frac{-v_0}{a}
\]  \hspace{1cm} (2.15)

the time interval from the moment the brakes are applied until the car comes to a complete stop. Substituting the values of the initial velocity \( v_0 \) and the constant acceleration \( a \) into equation 2.15, we have

\[
t_{\text{stop}} = \frac{-v_0}{a} = \frac{-26.4 \, \text{m/s}}{5.50 \, \text{m/s}^2} = 4.80 \, \text{s}
\]

It will take 4.80 s for the car to come to a stop if nothing gets in its way to change its rate of deceleration. Note how the units cancel in the equation until the final unit becomes seconds, that is,

\[
\frac{v_0}{a} = \frac{\text{m/s}}{\text{m/s}^2} = \frac{\text{1/s}}{\text{1/s}^2} = \frac{\text{1}}{\text{1/s}} = \text{1/s} = \frac{\text{s}}{\text{s}^2} = \text{1/s}
\]

Thus, \( (\text{m/s})/(\text{m/s}^2) \), comes out to have the unit seconds, which it must since it represents the time for the car to come to a stop.

Now that we know the time for the car to come to a stop, we can substitute that value back into equation 2.14 and find the distance the car will travel in the 4.80 s:
The car will come to a stop in 63.6 m. Since the tree is only 60.0 m in front of the car, it cannot come to a stop in time and will hit the tree.

In addition to the velocity and position of a moving body at any instant of time, we sometimes need to know the velocity of the moving body at a particular displacement \( x \). In the example of the car hitting the tree, we might want to know the velocity of the car when it hits the tree. That is, what is the velocity of the car when the displacement \( x \) of the car is equal to 60.0 m?

To find the velocity as a function of displacement \( x \), we must eliminate time from our kinematic equations. To do this, we start with equation 2.13 for the displacement of the moving body in terms of the average velocity,

\[
x = v_{\text{avg}} t = \left( \frac{v_0 + v}{2} \right) t
\]  

(2.13)

But \( v \) is the velocity of the moving body at any time \( t \), given by

\[
v = v_0 + at
\]  

(2.11)

Solving for \( t \) gives

\[
t = \frac{v - v_0}{a}
\]

Substituting this value into equation 2.13 gives

\[
x = \left( \frac{v_0 + v}{2} \right) t = \left( \frac{v_0 + v}{2} \right) \left( \frac{v - v_0}{a} \right)
\]

\[
= \left( \frac{v_0 + v}{2a} \right) (v - v_0)
\]

\[
2ax = v_0v + v^2 - v_0v - v_0^2
\]

\[
= v^2 - v_0^2
\]

Solving for \( v^2 \), we obtain the third kinematic equation,

\[
v^2 = v_0^2 + 2ax
\]  

(2.16)

which is used to determine the velocity \( v \) of the moving body at any displacement \( x \).
Let us now go back to the problem of the car moving down the road, with a tree lying in the road 60.0 m in front of the car. We already know that the car will hit the tree, but at what velocity will it be going when it hits the tree? That is, what is the velocity of the car at the displacement of 60.0 m? Using equation 2.16 with \( x = 60.0 \text{ m} \), \( v_0 = 26.4 \text{ m/s} \), and \( a = -5.50 \text{ m/s}^2 \), and solving for \( v \) gives

\[
v^2 = v_0^2 + 2ax
\]
\[
= (26.4 \text{ m/s})^2 + 2(-5.50 \text{ m/s}^2)(60 \text{ m})
\]
\[
= 697 \text{ m}^2/\text{s}^2 - 660 \text{ m}^2/\text{s}^2
\]
\[
= 37.0 \text{ m}^2/\text{s}^2
\]

\[
v = 6.08 \frac{\text{m}}{\text{s}} \left( \frac{3.60 \text{ km/hr}}{1 \text{ m/s}} \right)
\]

and finally,

\[
v = 21.9 \text{ km/hr}
\]

When the car hits the tree it will be moving at 21.9 km/hr, so the car may need a new bumper or fender. Equation 2.16 allows us to determine the velocity of the moving body at any displacement \( x \).

A problem similar to that of the car and the tree involves the maximum velocity that a car can move and still have adequate time to stop before hitting something the driver sees on the road in front of the car. Let us again assume that the car decelerates at the same constant rate as before, \( a = -5.50 \text{ m/s}^2 \), and that the low beam headlights of the car are capable of illuminating a 60.0 m distance of the road. Using equation 2.16, which gives the velocity of the car as a function of displacement, let us find the maximum value of \( v_0 \) such that \( v \) is equal to zero when the car has the displacement \( x \). That is,

\[
v^2 = v_0^2 + 2ax
\]
\[
0 = v_0^2 + 2ax
\]
\[
v_0 = \sqrt{-2ax}
\]
\[
= \sqrt{-2(-5.50 \text{ m/s}^2)(60.0 \text{ m})}
\]
\[
= \sqrt{660} \text{ m}^2/\text{s}^2
\]
\[
v = 25.7 \frac{\text{m}}{\text{s}} \left( \frac{3.60 \text{ km/hr}}{1 \text{ m/s}} \right)
\]
\[
= 92.5 \text{ km/hr}
\]

If the car decelerates at the constant rate of 5.50 m/s\(^2\) and the low beam headlights are only capable of illuminating a distance of 60.0 m, then the maximum safe velocity of the car at night without hitting something is 92.5 km/hr. For velocities faster than this, the distance it takes to bring the car to a stop is greater than the distance the driver can see with low beam headlights. If you see it, you’ll hit it! Of course these results are based on the assumption that the car decelerates...
at 5.50 m/s². This number depends on the condition of the brakes and tires and road conditions, and will be different for each car. To increase the maximum safe velocity of the car at night without hitting something, your car has high beam “bright” lights that illuminates a greater distance of the road. But even with these brighter beams, there is still another maximum safe driving speed, and if you drive faster than that, if you see it, you’ll hit it.

In summary, the three kinematic equations,

\[ x = v_0 t + \frac{1}{2} at^2 \]  
\[ v = v_0 + at \]  
\[ v^2 = v_0^2 + 2ax \]

are used to describe the motion of an object undergoing constant acceleration. The first equation gives the displacement of the object at any instant of time. The second equation gives the body’s velocity at any instant of time. The third equation gives the velocity of the body at any displacement x.

These equations are used for either positive or negative accelerations. Remember the three kinematic equations hold only for constant acceleration. If the acceleration varies with time then more advanced techniques must be used to determine the position and velocity of the moving object and will be discussed in section 2.10.

2.7 The Freely Falling Body

Another example of the motion of a body in one dimension is the freely falling body. A freely falling body is defined as a body that is moving freely under the influence of gravity, where it is assumed that the effect of air resistance is negligible. The body can have an upward, downward, or even zero initial velocity. The simplest of the freely falling bodies we discuss is the body dropped in the vicinity of the surface of the earth. That is, the first case to be considered is the one with zero initial velocity, \( v_0 = 0 \). The motion of a body in the vicinity of the surface of the earth with either an upward or downward initial velocity will be considered in section 2.9.

In chapter 5 on Newton’s second law of motion, we will see that whenever an unbalanced force \( F \) acts on an object of mass \( m \), it gives that object an acceleration, \( a \). The gravitational force that the earth exerts on an object causes that object to have an acceleration. This acceleration is called the acceleration due to gravity and is denoted by the letter \( g \). Therefore, any time a body is dropped near the surface of the earth, that body, ignoring air friction, experiences an acceleration \( g \). From experiments in the laboratory we know that the value of \( g \) near the surface of the earth is constant and is given by

\[ g \]
Any body that falls with the acceleration due to gravity, \( g \), is called a freely falling body.

Originally Aristotle said that a heavier body falls faster than a lighter body and on his authority this statement was accepted as truth for 1800 years. It was not disproved until the end of the sixteenth century when Simon Stevin (Stevinus) of Bruges (1548-1620) dropped balls of very different weights and found that they all fell at the same rate. That is, the balls were all dropped from the same height at the same time and all landed at the ground simultaneously. The argument still persisted that a ball certainly drops faster than a feather, but Galileo Galilei (1564-1642) explained the difference in the motion by saying that it is the air’s resistance that slows up the feather. If the air were not present, the ball and the feather would accelerate at the same rate.

A standard demonstration of the rate of fall is the penny and the feather demonstration. A long tube containing a penny and a feather is used, as shown in figure 2.15. If we turn the tube upside down, first we observe that the penny falls to the bottom of the tube before the feather. Then we connect the tube to a vacuum pump and evacuate most of the air from the tube. Again we turn the tube upside down, and now the penny and feather do indeed fall at the same rate and reach the bottom of the tube at the same time. Thus, it is the air friction that causes the feather to fall at the slower rate.

Another demonstration of a freely falling body, performed by the Apollo astronauts on the surface of the moon, was seen by millions of people on television. One of the astronauts dropped a feather and a hammer simultaneously and millions saw them fall at the same rate, figure 2.16. Remember, there is no atmosphere on the moon.

Therefore, neglecting air friction, all freely falling bodies accelerate downward at the same rate regardless of their mass. Recall that the acceleration of
a body was defined as the change in its velocity with respect to time, that is,

\[ a = \frac{\Delta v}{\Delta t} \] (2.7)

Hence, a body that undergoes an acceleration due to gravity of 9.80 m/s², has its velocity changing by 9.80 m/s every second. If we neglect the effects of air friction, every body near the surface of the earth accelerates downward at that rate, whether the body is very large or very small. For all of the problems considered in this chapter, we neglect the effects of air resistance. The effect of air resistance on motion will be treated in chapter 6.

Since the acceleration due to gravity is constant near the surface of the earth, we can determine the position and velocity of the freely falling body by using the kinematic equations 2.11, 2.14, and 2.16. However, because the motion is vertical, we designate the displacement by \( y \) in the kinematic equations:

\[ v = v_0 + at \] (2.11)

\[ y = v_0 t + \frac{1}{2} at^2 \] (2.14)

\[ v^2 = v_0^2 + 2ay \] (2.16)

Since the first case we consider is a body that is dropped, we will set the initial velocity \( v_0 \) equal to zero in the kinematic equations. Also the acceleration of the moving body is now the acceleration due to gravity, therefore we write the acceleration as

\[ a = -g \] (2.17)
The minus sign in equation 2.17 is consistent with our previous convention for one-dimensional motion. Motion in the direction of the positive axis is considered positive, while motion in the direction of the negative axis is considered negative. Hence, all quantities in the upward direction (positive $y$-direction) are considered positive, whether displacements, velocities, or accelerations. And all quantities in the downward direction (negative $y$-direction) are considered negative, whether displacements, velocities, or accelerations. The minus sign indicates that the direction of the acceleration is down, toward the center of the earth. This notation will be very useful later in describing the motion of projectiles. Therefore, the kinematic equations for a body dropped from rest ($v_0 = 0$) near the surface of the earth are

\[ y = -\frac{1}{2} gt^2 \]  
\[ v = -gt \]  
\[ v^2 = -2gy \]

Equation 2.18 gives the height or location of the freely falling body at any time, equation 2.19 gives its velocity at any time, and equation 2.20 gives the velocity of the freely falling body at any height $y$. This sign convention gives a negative value for the displacement $y$, which means that the zero position of the body is the position from which the body is dropped, and the body's location at any time $t$ will always be below that point. The minus sign on the velocity indicates that the direction of the velocity is downward.

Equations 2.18, 2.19, and 2.20 completely describe the motion of the freely falling body that is dropped from rest. As an example, let us calculate the distance fallen and velocity of a freely falling body as a function of time for the first 5 s of its fall. The results of the computations are written in figure 2.17. At $t = 0$ the body is located at $y = 0$, (top of figure 2.17) and its velocity is zero. We then release the body. Where is it at $t = 1$ s? Using equation 2.18, $y_1$ is the displacement of the body (distance fallen) at the end of 1 s:

\[ y_1 = -\frac{1}{2} gt^2 = -\frac{1}{2} (9.80 \text{ m/s}^2)(1 \text{ s})^2 = -4.90 \text{ m} \]

The minus sign indicates that the body is 4.90 m below the starting point. To find the velocity at the end of 1 s, we use equation 2.19:

\[ v_1 = -gt = (-9.80 \text{ m/s}^2)(1 \text{ s}) = -9.80 \text{ m/s} \]

The velocity is 9.80 m/s downward at the end of 1 s. The position and velocity at the end of 1 s are shown in figure 2.17. For $t = 2$ s, the displacement and velocity are

\[ y_2 = -\frac{1}{2} gt^2 = -\frac{1}{2} (9.80 \text{ m/s}^2)(2 \text{ s})^2 = -19.6 \text{ m} \]
\[ v_2 = -gt = (-9.80 \text{ m/s}^2)(2 \text{ s}) = -19.6 \text{ m/s} \]

\[ y = \frac{-1}{2} gt^2 = \frac{-1}{2} (9.80 \text{ m/s}^2)(3 \text{ s})^2 = -44.1 \text{ m} \]

\[ v_3 = -gt = (-9.80 \text{ m/s}^2)(3 \text{ s}) = -29.4 \text{ m/s} \]

At the end of 3 s the body has fallen a distance of 88.2 m and is moving downward at a velocity of 29.4 m/s.

The distance and velocity for \( t = 4 \text{ s} \) and \( t = 5 \text{ s} \) are found similarly and are shown in figure 2.17. One of the first things to observe in figure 2.17 is that an object falls a relatively large distance in only a few seconds of time. Also note that the object does not fall equal distances in equal times, but rather the distance interval becomes greater for the same time interval as time increases. This is, of course, the result of the \( t^2 \) in equation 2.18 and is a characteristic of accelerated motion. Also note that the change in the velocity in any 1-s time interval is 9.80 m/s, which is exactly what we meant by saying the acceleration due to gravity is 9.80 m/s².

We stated previously that the average velocity during a time interval is exactly equal to the instantaneous value of the velocity at the exact center of that time interval. We can see that this is the case by inspecting figure 2.17. For example, if we take the time interval as between \( t = 3 \text{ s} \) and \( t = 5 \text{ s} \), then the average velocity between the third and fifth second is

\[ v_{35\text{avg}} = \frac{v_5 + v_3}{2} = \frac{-49.0 \text{ m/s} + (-29.4 \text{ m/s})}{2} = \frac{-78.4 \text{ m/s}}{2} = -39.2 \text{ m/s} = v_4 \]
The average velocity between the time interval of 3 and 5 s, \( v_{35\text{avg}} \), is exactly equal to \( v_4 \), the instantaneous velocity at \( t = 4 \) seconds, which is the exact center of the 3-5 time interval, as we can see in figure 2.17. The figure also shows the characteristic of an average velocity. At the beginning of the time interval the actual velocity is less than the average value, while at the end of the time interval the actual velocity is greater than the average value, but right at the center of the time interval the actual velocity is equal to the average velocity. Note that the average velocity occurs at the center of the time interval and not the center of the space interval.

In summary, we can see the enormous power inherent in the kinematic equations. An object was dropped from rest and the kinematic equations completely described the position and velocity of that object at any instant of time. All that information was contained in those equations.

**Example 2.7**

*Using the kinematic equation for free fall.* A student's book falls out the window of the physics laboratory. How long does it take to fall to the ground 20.0 m below? With what velocity does the book hit the ground?

**Solution**

To find the time for the book to fall to the ground we solve equation 2.18 for the time \( t \) as

\[
t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2(-20.0 \text{ m})}{9.80 \text{ m/s}^2}} = 2.02 \text{ s}
\]

Notice that the distance of 20.0 m is written as a negative number because the ground is 20.0 m below the point from which the book starts to fall. That is, the ground is on the negative \( y \)-axis at \( y = -20.0 \text{ m} \).

The velocity of the book as it hits the ground is found from equation 2.19 as

\[
v = -gt = -(9.80 \text{ m/s}^2)(2.02 \text{ s}) = -19.8 \text{ m/s}
\]

Notice that the answer is negative, indicating that the velocity is in the negative \( y \)-direction.
2.8 Determination of Your Reaction Time by a Freely Falling Body

How long a period of time does it take for you to react to something? How can you measure this reaction time? It would be very difficult to use a clock to measure reaction time because it will take some reaction time to turn the clock on and off. However, a freely falling body can be used to measure reaction time. To see how this is accomplished, have one student hold a vertical meter stick near the top, as shown in figure 2.18(a). The second student then places his or her hand at the zero

![Figure 2.18 Measurement of reaction time.](image)

of the meter stick (the bottom of the stick) with thumb and forefinger extended. The thumb and forefinger should be open about 3 to 5 cm. When the first student drops the meter stick, the second student catches it with the thumb and finger [figure 2.18(b)].

As the meter stick is released, it becomes a freely falling body and hence falls a distance $y$ in a time $t$:

$$y = -\frac{1}{2}gt^2$$
The location of the fingers on the meter stick, where the meter stick was caught, represents the distance \( y \) that the meter stick has fallen. Solving for the time \( t \) we get

\[
t = \sqrt{\frac{2y}{g}}
\]  

(2.21)

Since we have measured \( y \), the distance the meter stick has fallen, and we know the acceleration due to gravity \( g \), we can do the simple calculation in equation 2.21 and determine your reaction time. (Remember that the value of \( y \) placed into equation 2.21 will be a negative number and hence we will take the square root of a positive quantity since the square root of a negative number is not defined.)

If you practice catching the meter stick, you will be able to catch it in less time. But eventually you reach a time that, no matter how much you practice, you cannot make smaller. This time is your minimum reaction time--the time it takes for your eye to first see the stick drop and then communicate this message to your brain. Your brain then communicates this information through nerves and muscles to your fingers and then you catch the stick. Your normal reaction time is most probably the time that you first caught the stick. A normal reaction time to catch the meter stick is about 0.2 to 0.3 seconds.

Note that this is not quite the same reaction time it would take to react to a red light while driving a car, because in that case, part of the communication from the brain would entail lifting your leg from the accelerator, placing it on the brake pedal, and then pressing. The brake pedal transmits this force to the brake fluid, which transmits this force to the brakes, which now initiates the deceleration process. The motion of more muscles and mass would consequently take a longer period of time. A normal reaction time in a car is approximately 0.5 s. To obtain a more accurate value of the stopping distance for a car we also need to include the distance that the car moves while the driver reacts to the red light.

**Example 2.8**

Measuring your reaction time. One student holds a vertical meter stick near the top and then drops it. The second student then catches it after the stick has fallen 23.5 cm. Using the kinematic equation for free fall, determine the reaction time of the second student.

**Solution**

Your reaction time is the time it takes you to react to something. For the falling meterstick, it is the time from the moment you see the meterstick drop, to the time you catch the meterstick. The falling meterstick is a freely falling body and to find the time for the meterstick to fall, we solve equation 2.21 for the time \( t \) as
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\[ t = \sqrt{\frac{2y}{g}} \]
\[ = \sqrt{\frac{2(-0.235 \text{ m})}{9.80 \text{ m/s}^2}} \]
\[ = 0.419 \text{ s} \]

To go to this interactive example click on this sentence.

2.9 Projectile Motion in One Dimension

A case one step more general than the freely falling body dropped from rest, is the motion of a body that is thrown up or down with an initial velocity \( v_0 \) near the surface of the earth. This type of motion is called **projectile motion** in one dimension. Remember, however, that this type of motion still falls into the category of a freely falling body because the object experiences the acceleration \( g \) downward throughout its motion. The kinematic equations for projectile motion are

\[
\begin{align*}
y &= v_0 t - \frac{1}{2} gt^2 \\
v &= v_0 - gt \\
v^2 &= v_0^2 - 2gy
\end{align*}
\]  

(2.22) (2.23) (2.24)

These three equations completely describe the motion of a projectile in one dimension. Note that these equations are more general than those for the body dropped from rest because they contain the initial velocity \( v_0 \). In fact, if \( v_0 \) is set equal to zero these equations reduce to the ones studied for the body dropped from rest.

In the previous cases of motion, we were concerned only with motion in one direction. Here there are two possible directions, up and down. According to our convention the upward direction is positive and the downward direction is negative. Hence, if the projectile is initially thrown upward, \( v_0 \) is positive; if the projectile is initially thrown downward, \( v_0 \) is negative. Also note that whether the projectile is thrown up or down, the acceleration due to gravity always points downward. If it did not, then a ball thrown upward would continue to rise forever and would leave the earth, a result that is contrary to observation.

Let us now consider the motion of a projectile thrown upward. Figure 2.19 shows its path through space, which is called a **trajectory**. The projectile goes straight up, and then straight down. The downward portion of the motion is slightly displaced from the upward portion to clearly show the two different parts of the motion. For example, suppose the projectile is a baseball thrown straight upward with an initial velocity \( v_0 = 30.0 \text{ m/s} \). We want to determine
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1. The maximum height of the ball.
2. The time it takes for the ball to rise to the top of its trajectory.
3. The total time that the ball is in the air.
4. The velocity of the ball as it strikes the ground.
5. The position and velocity of the ball at any time \( t \), for example, for \( t = 4.00 \) s.

We are asking for a great deal of information, especially considering that the only data given is the initial position and velocity of the ball. Yet all this information can be obtained using the three kinematic equations 2.22, 2.23, and 2.24. In fact, any time we try to solve any kinematic problem, the first thing is to write down the kinematic equations, because somehow, somewhere, the answers are in those equations. It is just a matter of manipulating them into the right form to obtain the information we want about the motion of the projectile.

Let us now solve the problem of projectile motion in one dimension.

**Find the Maximum Height of the Ball**

Equation 2.22 tells us the height of the ball at any instant of time, and we could use it to find the maximum height if we knew the time for the projectile to rise to the top of its trajectory. But at this point that time is unknown, (In fact, that is what we are asking for in question 2.) so we cannot use equation 2.22. Equation 2.24 tells us the velocity of the moving body at any height \( y \). The velocity of the ball is positive on the way up, and negative on the way down, so therefore it must have gone through zero somewhere. In fact, the velocity of the ball is zero when the ball is at the very top of its trajectory. If it were greater than zero the ball would continue to rise, if it were less than zero the ball would be on its way down. Therefore, at the top of the trajectory, the position of maximum height, \( v = 0 \), and equation 2.24,

\[
  v^2 = v_0^2 - 2gy
\]

becomes

\[
  0 = v_0^2 - 2gy_{\text{max}}
\]

where \( y_{\text{max}} \) is the maximum height of the projectile. For any other height \( y \), the velocity is either positive indicating that the ball is on its way up, or negative.
indicating that it is on the way down. Solving for \( y_{\text{max}} \), the maximum height of the ball is

\[
2gy_{\text{max}} = v_0^2
\]

\[
y_{\text{max}} = \frac{v_0^2}{2g}
\]  \hspace{1cm} (2.25)

Inserting numbers into equation 2.25, we get

\[
y_{\text{max}} = \frac{v_0^2}{2g}
\]

\[
= \frac{(30.0 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)}
\]

\[
= 45.9 \text{ m}
\]

The ball will rise to a maximum height of 45.9 m.

**Find the Time for the Ball to Rise to the Top of the Trajectory**

We have seen that when the projectile is at the top of its trajectory, \( v = 0 \). Therefore, equation 2.23,

\[
v = v_0 - gt
\]

becomes

\[
0 = v_0 - gt_r
\]

where \( t_r \) is the time for the projectile to rise to the top of its path. Only at this value of time does the velocity equal zero. At any other time the velocity is either positive or negative, depending on whether the ball is on its way up or down. Solving for \( t_r \) we get

\[
t_r = \frac{v_0}{g}
\]  \hspace{1cm} (2.26)

the time for the ball to rise to the top of its trajectory. Inserting numbers into equation 2.26 we obtain

\[
t_r = \frac{v_0}{g} = \frac{30.0 \text{ m/s}}{9.80 \text{ m/s}^2}
\]

\[
= 3.06 \text{ s}
\]

It takes 3.06 s for the ball to rise to the top of the trajectory. Notice that the ball has the acceleration \(-g\) at the top of the trajectory even though the velocity is zero at that instant. That is, in any kind of motion, we can have a nonzero acceleration even though the velocity is zero. The important thing for an acceleration is the *change in velocity*, not the velocity itself. At the top, the change in velocity is not zero, because the velocity is changing from positive values on the way up, to negative values on the way down.

The time \( t_r \) could also have been found using equation 2.22,
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\[ y = v_0 t - \frac{1}{2} gt^2 \]

by substituting the maximum height of 45.9 m for \( y \). Even though this also gives the correct solution, the algebra and arithmetic are slightly more difficult because a quadratic equation for \( t \) would have to be solved. (We will show how the quadratic equation is used in the solution of projectile motion in chapter 4.)

**Find the Total Time that the Object Is in the Air**

When \( t \) is equal to the total time \( t_t \), that the projectile is in the air, \( y \) is equal to zero. That is, during the time from \( t = 0 \) to \( t = t_t \), the projectile goes from the ground to its maximum height and then falls back to the ground. Using equation 2.22, the height of the projectile at any time \( t \),

\[ y = v_0 t - \frac{1}{2} gt^2 \]

with \( t = t_t \) and \( y = 0 \), we get

\[ 0 = v_0 t_t - \frac{1}{2} gt_t^2 \tag{2.27} \]

Solving for \( t_t \) we obtain

\[ t_t = \frac{2v_0}{g} \tag{2.28} \]

the total time that the projectile is in the air. Recall from equation 2.26 that the time for the ball to rise to the top of its trajectory is \( t_r = \frac{v_0}{g} \). And the total time, equation 2.28, is just twice that value. Therefore, the total time that the projectile is in the air becomes

\[ t_t = \frac{2v_0}{g} = 2t_r \tag{2.29} \]

The total time that the projectile is in the air is twice the time it takes the projectile to rise to the top of its trajectory. Stated in another way, the time for the ball to go up to the top of the trajectory is equal to the time for the ball to come down to the ground.

For this particular problem,

\[ t_t = 2t_r = 2(3.06 \text{ s}) = 6.12 \text{ s} \]

The projectile will be in the air for a total of 6.12 s. Also note that equation 2.27 is really a quadratic equation with two roots. One of which we can see by inspection is \( t = 0 \), which is just the initial moment that the ball is launched.
**Find the Velocity of the Ball as It Strikes the Ground**

There are two ways to find the velocity of the ball at the ground. The simplest is to use equation 2.24,

\[ v^2 = v_0^2 - 2gy \]

noting that the height is equal to zero \((y = 0)\) when the ball is back on the ground. Therefore,

\[ v_g^2 = v_0^2 \]

and

\[ v_g = \pm v_0 \quad (2.30) \]

The two roots represent the velocity at the two times that \(y = 0\), namely, when the ball is first thrown up \((t = 0)\), with an initial velocity \(+v_0\), and when the ball lands \((t = t_l)\) with a final velocity of \(-v_0\) (the minus sign indicates that the ball is on its way down).

Another way to find the velocity at the ground is to use equation 2.23,

\[ v = v_0 - gt \]

which represents the velocity of the projectile at any instant of time. If we let \(t\) be the total time that the projectile is in the air (i.e., \(t = t_l\)), then \(v = v_g\), the velocity of the ball at the ground. Thus,

\[ v_g = v_0 - gt_l \quad (2.31) \]

But we have already seen that

\[ t_l = \frac{2v_o}{g} \quad (2.28) \]

Substituting equation 2.28 into equation 2.31 gives

\[ v_g = v_0 - g\left(\frac{2v_0}{g}\right) \]

Hence,

\[ v_g = -v_0 \]

The velocity of the ball as it strikes the ground is equal to the negative of the original velocity, with which the ball was thrown upward, that is,

\[ v_g = -v_0 = -30.0 \text{ m/s} \]

**Find the Position and Velocity of the Ball at \(t = 4.00 \text{ s}\)**

The position of the ball at any time \(t\) is given by equation 2.22 as
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\[ y = v_0t - \frac{1}{2}gt^2 \]

Substituting in the values for \( t = 4.00 \text{ s} \) gives

\[ y_4 = (30.0 \text{ m/s})(4.00 \text{ s}) - \frac{1}{2}(9.80 \text{ m/s}^2)(4.00 \text{ s})^2 \]

\[ = 120 \text{ m} - 78.4 \text{ m} \]

\[ = 41.6 \text{ m} \]

At \( t = 4.00 \text{ s} \) the ball is 41.6 m above the ground.

The velocity of the ball at any time is given by equation 2.23 as

\[ v = v_0 - gt \]

For \( t = 4.00 \text{ s} \), the velocity becomes

\[ v_4 = 30.0 \text{ m/s} - (9.80 \text{ m/s}^2)(4.00 \text{ s}) \]

\[ = 30.0 \text{ m/s} - 39.2 \text{ m/s} \]

\[ = -9.2 \text{ m/s} \]

At the end of 4 s the velocity of the ball is \(-9.2 \text{ m/s}\), where the negative sign indicates that the ball is on its way down. We could have used equation 2.22 for every value of time and plotted the entire trajectory, as shown in figure 2.20.

There is great beauty and power in these few simple equations, because with them we can completely predict the motion of the projectile for any time, simply by knowing its initial position and velocity. This is a characteristic of the field of physics. First we observe how nature works. Then we make a mathematical model of nature in terms of certain equations. We manipulate these equations until we can make a prediction, and this prediction yields information that we usually have no other way of knowing.

For example, how could you know that the velocity of the ball after 4.00 s would be \(-9.2 \text{ m/s}\)? In general, there is no way of knowing that. Yet we have actually captured a small piece of nature in our model and have seen how it works.
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**Figure 2.20** Results of projectile motion in one dimension.

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### Example 2.9

A projectile is fired straight up from the top of a building. A projectile is fired from the top of a building at an initial velocity of 35.0 m/s upward. The top of the building is 30.0 m above the ground. The motion is shown in figure 2.21. Find (a) the maximum height of the projectile, (b) the time for the projectile to reach its maximum height, (c) the velocity of the projectile as it strikes the ground, and (d) the total time that the projectile is in the air.

**Solution**

We will solve this problem using the techniques just developed.

---

**Figure 2.21** A projectile is fired vertically from the top of a building.

**a.** To find the maximum height of the projectile we again note that at the top of the trajectory \( v = 0 \). Using equation 2.24,

\[
v^2 = v_0^2 - 2gy
\]

and setting \( v = 0 \) we obtain

\[
0 = v_0^2 - 2gy_{\text{max}}
\]

Solving for the maximum height,

\[
y_{\text{max}} = \frac{v_0^2}{2g} = \frac{(35.0 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} = 62.5 \text{ m}
\]

The projectile's maximum height is 62.5 m above the roof of the building, or 92.5 m above the ground.

**b.** To find the time for the projectile to reach its maximum height we again note that at the maximum height \( v = 0 \). Substituting \( v = 0 \) into equation 2.23, we get
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\[ 0 = v_0 - gt_r \]

Solving for the time to rise to the top of the trajectory, we get

\[ t_r = \frac{v_0}{g} = \frac{35.0 \text{ m/s}}{9.80 \text{ m/s}^2} = 3.57 \text{ s} \]

It takes 3.57 s for the ball to rise from the top of the roof to the top of its trajectory.

c. To find the velocity of the projectile when it strikes the ground, we use equation 2.24. When \( y = -30.0 \text{ m} \) the projectile will be on the ground, and its velocity as it strikes the ground is

\[
\begin{align*}
v^2 &= v_0^2 - 2gy \\
(v_g)^2 &= (35.0 \text{ m/s})^2 - 2(9.80 \text{ m/s}^2)(-30.0 \text{ m}) \\
&= 1225 \text{ m}^2/\text{s}^2 + 588 \text{ m}^2/\text{s}^2 = 1813 \text{ m}^2/\text{s}^2 \\
v_g &= -42.6 \text{ m/s}
\end{align*}
\]

The projectile hits the ground at a velocity of \(-42.6 \text{ m/s}\). Note that this value is greater than the initial velocity \(v_0\), because the projectile does not hit the roof on its way down, but rather hits the ground 30.0 m below the level of the roof. The acceleration has acted for a longer time, thereby imparting a greater velocity to the projectile.

d. To find the total time that the projectile is in the air we use equation 2.23,

\[ v = v_0 - gt \]

But when \( t \) is equal to the total time that the projectile is in the air, the velocity is equal to the velocity at the ground (i.e., \( v = v_g \)). Therefore,

\[ v_g = v_0 - g t_t \]

Solving for the total time, we get

\[ t_t = \frac{v_0 - v_g}{g} = \frac{35.0 \text{ m/s} - (-42.6 \text{ m/s})}{9.80 \text{ m/s}^2} = \frac{(35.0 + 42.6)\text{m/s}}{9.80 \text{ m/s}^2} = 7.92 \text{ s} \]

The total time that the projectile is in the air is 7.92 s. Note that it is not twice the time for the projectile to rise because the projectile did not return to the level where it started from, but rather to 30.0 m below that level.

To go to this interactive example click on this sentence.
*2.10 The Calculus and Kinematics*

Up to now our analysis of kinematics has been essentially graphical and algebraic. A more sophisticated approach to the analysis of motion is obtained by using the calculus. We start with the motion of an accelerated body. The definition of the acceleration of the moving body is given by

$$ a = \frac{dv}{dt} \quad (2.32) $$

and we rewrite it in the following form

$$ dv = adt \quad (2.33) $$

Let us now integrate equation 2.33

$$ \int_{v_0}^{v} dv = \int_{0}^{t} adt \quad (2.34) $$

At the beginning of the motion the time is $t = 0$ and this becomes the lower limit in the second integral. At this same time the velocity of the moving body is $v_0$, which becomes the lower limit in the first integral. At the later time $t$, the upper limit in the second integral, the velocity of the moving body is $v$, which becomes the upper limit in the first integral. Evaluating the integral on the left we obtain

$$ v - v_0 = - \int_{0}^{t} v \, dt \quad (2.35) $$

and equation 2.34 becomes

$$ v - v_0 = \int_{0}^{t} adt \quad (2.36) $$

In general the acceleration term $a$ in equation 2.36 could vary with time. If it does we would have to know that functional relation before we can solve equation 2.36. On the other hand, if the acceleration of the moving body is a constant, then it can be taken out from under the integral sign to obtain

$$ v - v_0 = \int_{0}^{t} adt = a \int_{0}^{t} dt = at \quad (2.37) $$

or

$$ v - v_0 = at \quad (2.38) $$

$$ v = v_0 + at \quad (2.38) $$
Equation 2.38 gives the velocity of the moving body at any instant of time, when the acceleration of the body is a constant. Notice that this is the same equation we obtained in equation 2.11.

The velocity at any instant of time was given by equation 2.8 as

\[ v = \frac{dx}{dt} \]

Setting equation 2.8 equal to equation 2.38 gives

\[ v = \frac{dx}{dt} = v_0 + at \]

which becomes

\[ dx = (v_0 + at)dt \]

Upon integrating

\[ \int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at) dt \]

At the time \( t = 0 \), the lower limit in the second integral, the displacement of the moving body is \( x_0 \), which becomes the lower limit in the first integral. At the time \( t = t \), the upper limit in the second integral, the displacement of the moving body is \( x \), which becomes the upper limit in the first integral. Separating the two terms on the right into two separate integrals we get

\[ \int_{x_0}^{x} dx = \int_{0}^{t} v_0 dt + \int_{0}^{t} at dt \]

Since \( v_0 \) and \( a \) are constant they come out of the integral sign to yield

\[ x - x_0 = v_0 \int_{0}^{t} dt + a \int_{0}^{t} t dt = v_0 t + \frac{1}{2} at^2 \]

or

\[ x = x_0 + v_0 t + \frac{1}{2} at^2 \]

Equation 2.39 gives the displacement \( x \) of the moving body at any instant of time for a body that is undergoing constant acceleration. Equation 2.39 looks like equation 2.14 except for the term \( x_0 \), which represents the position of the moving body at the time \( t = 0 \). In our derivation of equation 2.14, we assumed that the body was at the origin of our reference system and hence \( x_0 = 0 \) in that equation. Equation 2.39 is more general in that it allows for the moving body to be at some place other than the origin at \( t = 0 \).

In summary equation 2.38 gives the velocity of the moving body at any time \( t \) while equation 2.39 gives the displacement of the moving body at any time \( t \).
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\[ v = v_0 + at \]  \hspace{1cm} (2.38)
\[ x = x_0 + v_0t + \frac{1}{2}at^2 \]  \hspace{1cm} (2.39)

The term \( x_0 \) represents the initial coordinates of the body at the time \( t = 0 \). In most of the analysis we did so far we assumed that the moving body was at the origin of the coordinate system at the time \( t = 0 \) and therefore we set \( x_0 = 0 \) and it was not found in our kinematic equations. For the case of projectile motion we can let \( x = y \) and \( a = -g \) and we then obtain the equations for projectile motion 2.22, 2.23.

It may seem strange to introduce the calculus here at the end of the chapter to obtain the same equations we obtained algebraically. Why go to all the trouble to learn calculus if the results are the same as we obtained using algebra? Why not just use algebra to do physics? The reason becomes evident if you look at equation 2.37. The acceleration term \( a \) is under the integral sign. We assumed that \( a \) was a constant and pulled it out from under the integral sign and that allowed us to solve equation 2.37 to obtain the standard kinematic equations.

But what happens if the acceleration \( a \) is not a constant? The kinematic equations were derived on the assumption that \( a \) is equal to a constant; if is not a constant, those equations no longer represent the physical problem and are now essentially useless. This is a characteristic of physics that we will see over and over as we proceed through our study of physics. In deriving an equation certain assumptions are made. If those assumptions hold in the application of the equation, then the equation is valid. If the assumptions do not hold, then the equation is no longer valid.

As an example let us now solve a problem in which the acceleration is not a constant. A body moves in the \( x \)-direction with an acceleration that varies with time according to the relation

\[ a = A + Bt \]  \hspace{1cm} (2.40)

where \( A \) is a constant equal to 4.00 m/s\(^2\) and \( B \) is another constant equal to 0.300 m/s\(^3\). Find the equations for the velocity and displacement of the moving body at any time \( t \). In particular, if the initial displacement of the moving body is \( x_0 = 0 \) and the initial velocity \( v_0 = 2.00 \) m/s, find the numerical values of the velocity, displacement, and acceleration at \( t = 5.00 \) s. The velocity is found from equation 2.36 as

\[ v - v_0 = \int_0^t adt \]  \hspace{1cm} (2.41)

Placing the value of the acceleration into equation 2.41 yields

\[ v - v_0 = \int_0^t (A + Bt) dt \]  \hspace{1cm} (2.42)
\[ v - v_0 = \int_0^t A dt + \int_0^t B dt \]

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\[ v - v_0 = At + \frac{Bt^2}{2} \]

Solving for \( v \) we get the velocity of the moving body at any instant of time as

\[ v = v_0 + At + \frac{Bt^2}{2} \]  \hspace{1cm} (2.43)

The displacement of the moving body is now found from equation 2.43 as

\[ v = \frac{dx}{dt} = v_0 + At + \frac{Bt^2}{2} \]

\[ dx = \left( v_0 + At + \frac{Bt^2}{2} \right) dt \]

Upon integrating

\[ \int_{x_0}^{x} dx = \int_{0}^{t} \left( v_0 + At + \frac{Bt^2}{2} \right) dt = \int_{0}^{t} v_0 dt + \int_{0}^{t} At dt + \int_{0}^{t} \frac{Bt^2}{2} dt \]

\[ x - x_0 = v_0 t + \frac{At^2}{2} + \frac{Bt^3}{6} \]

\[ x = x_0 + v_0 t + \frac{At^2}{2} + \frac{Bt^3}{6} \]  \hspace{1cm} (2.44)

Equation 2.44 gives the displacement of the moving body at any instant of time.

For the specific initial conditions we are given, the velocity at 5.00 s is found from equation 2.43 to be

\[ v = v_0 + At + \frac{1}{2} Bt^2 \]

\[ v_5 = 2.00 \text{ m/s} + (4.00 \text{ m/s}^2)(5.00 \text{ s}) + \frac{1}{2} (0.300 \text{ m/s}^3)(5.00 \text{ s})^2 \]

\[ v_5 = 2.00 \text{ m/s} + 20.0 \text{ m/s} + 3.75 \text{ m/s} \]

\[ v_5 = 25.8 \text{ m/s} \]

The displacement of the moving body at 5 s is found from equation 2.44 as

\[ x = x_0 + v_0 t + \frac{1}{2} At^2 + \frac{1}{6} Bt^3 \]

\[ x_5 = 0 + (2.00 \text{ m/s})(5.00 \text{ s}) + \frac{1}{2} (4.00 \text{ m/s}^2)(5.00 \text{ s})^2 + \frac{1}{6} (0.300 \text{ m/s}^3)(5.00 \text{ s})^3 \]

\[ x_5 = 10.0 \text{ m} + 50.0 \text{ m} + 6.25 \text{ m} \]

\[ x_5 = 66.3 \text{ m} \]

The acceleration of the moving body was given in equation 2.40 as

\[ a = A + Bt \]

and its value at 5.00 s is

\[ a = A + Bt \]
The above problem was an example of variable acceleration. Equations 2.43 and 2.44 give the velocity and displacement for the moving body that has the variable acceleration $a = A + Bt$. Notice how these equations are very different than the kinematic equation 2.11 for velocity ($v = v_0 + at$) under constant acceleration, and the kinematic equation 2.14 for displacement ($x = v_0t + \frac{1}{2} at^2$) when the body was moving at constant velocity. If the body moves with a different acceleration, say $a = A \cos t$, then equation 2.43 and 2.44 cannot be used. For every different form of the acceleration $a$, there will be a different set of equations for the velocity and displacement. The procedure just shown will have to be followed to obtain the velocity and displacement of the moving body.

It should now be obvious that the kinematic equations, 2.38 and 2.39, hold only for motion at constant acceleration. Fortunately, most of the physical problems that we will encounter in this course will be for motion at constant acceleration and the standard form of the kinematic equations can be used. For those cases where the acceleration is variable, the procedure above must be followed.

As a final use of the calculus to solve kinematic problem let us consider the converse problem. That is, let us assume that we are given the displacement of the moving body as a function of time and we are asked to determine the velocity and acceleration of the body. As an example, the displacement of a certain moving body is given by

$$x = v_0t + At^2 + Bt^3$$

where $A$ and $B$ are constants. Find the equation for the velocity and acceleration of the moving body for any instant of time $t$. The equation for the velocity is found by differentiating equation 2.45. That is,

$$v = \frac{dx}{dt} = \frac{d}{dt}(v_0t + At^2 + Bt^3)$$

$$v = v_0 + 2At + 3Bt^2$$

The equation for the acceleration as a function of time is found by differentiating equation 2.46 as

$$a = \frac{dv}{dt} = \frac{d}{dt}(v_0 + 2At + 3Bt^2)$$

$$a = 0 + 2A + 6Bt$$

Hence using the calculus will allow us to solve more general and sometimes more difficult problems.
Have You Ever Wondered . . . ?
An Essay on the Application of Physics
Kinematics and Traffic Congestion

Have you ever wondered, while sitting in heavy traffic on the expressway, as shown in figure 1, why there is so much traffic congestion? The local radio station tells you there are no accidents on the road; the traffic is heavy because of volume. What does that mean? Why can’t cars move freely on the expressway? Why call it an expressway, if you have to move so slowly?

Figure 1 Does your highway look like this?

Let us apply some physics to the problem to help understand it. In particular, we will make a simplified model to help analyze the traffic congestion. In this model, we assume that the total length of the expressway $L$ is 10,000 ft (approximately two miles), the length of the car $x_c$ is 10 ft, and the speed of the car $v_0$ is 55 mph. How many cars of this size can safely fit on this expressway if they are all moving at 55 mph?

First, we need to determine the safe distance required for each car. If the car is moving at 55 mph (80.7 ft/s), and the car is capable of decelerating at $-18.0$ ft/s$^2$, the distance required to stop the car is found from equation 2.16,

$$v^2 = v_0^2 + 2ax$$

by noting that $v = 0$ when the car comes to a stop. Solving for the distance $x_a$ that the car moves while decelerating to a stop we get

$$x_a = \frac{-v_0^2}{2a} = \frac{-80.7^2}{2(-18)} \approx 181 \text{ ft}$$

We will depart from our custom of using only SI units here because most students will have a better feel for this discussion if it is done in the British engineering system of units.

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3\textsuperscript{3}We will depart from our custom of using only SI units here because most students will have a better feel for this discussion if it is done in the British engineering system of units.
Before the actual deceleration, the car will move, during the reaction time, a distance \(x_R\) given by

\[ x_R = v_0 t_R = (80.7 \text{ ft/s})(0.500 \text{ s}) = 40.4 \text{ ft} \]

where we assume that it takes the driver 0.500 s to react. The total distance \(\Delta L\) needed for each car on the expressway to safely come to rest is equal to the sum of the distance taken up by the car itself \(x_c\), the distance the car moves during the drivers reaction time \(x_R\), and the distance the car moves while it is decelerating \(x_d\). That is

\[ \Delta L = x_c + x_R + x_d = 10 \text{ ft} + 40.4 \text{ ft} + 181 \text{ ft} = 231 \text{ ft} \]

Because it takes a safe distance \(\Delta L\) for one car to come to rest, \(N\) cars will take a distance of \(N\Delta L\). The total length of the road \(L\) can then hold \(N\) cars, each requiring a distance \(\Delta L\) to stop, as seen in figure 2. Stated mathematically this is

\[ L = N\Delta L \]

Therefore, the number of cars \(N\) that can safely fit on this road is

\[ N = \frac{L}{\Delta L} = \frac{10,000 \text{ ft}}{231 \text{ ft}} = 43 \text{ cars} \]

Thus for a road 10,000 ft long, only 43 cars can fit safely on it when each is moving at 55 mph. If the number of cars on the road doubles, then the safe distance per car \(\Delta L\) must be halved because the product of \(N\) and \(\Delta L\) must equal \(L\) the total length of the road, which is a constant. Rewriting equation H2.1 in the form

\[ \Delta L = \frac{L}{N} = x_c + x_R + x_d = \frac{L}{N} \]

\[ x_c + v_0 t_R - \frac{v_0^2}{2a} = \frac{L}{N} \]

\[ (H2.2) \]
Notice in equation H2.2 that if the number of cars $N$ increases, the only thing that can change on this fixed length road is the initial velocity $v_0$ of each car. That is, by increasing the number of cars on the road, the velocity of each car must decrease, in order for each car to move safely. Equation H2.2 can be written in the quadratic form

$$\frac{-v_0^2 + v_0 t_R + x_c - L}{2a} = 0$$

which can be solved quadratically to yield

$$v_0 = at_R \pm \sqrt{(at_R)^2 + 2a \left( x_c - \frac{L}{N} \right)}$$

(H2.3)

Equation H2.3 gives the maximum velocity that $N$ cars can safely travel on a road $L$ ft long. (Don’t forget that $a$ is a negative number.) Using the same numerical values of $a$, $t_R$, $x_c$, and $L$ as above, equation H2.3 is plotted in figure 3 to show the safe velocity (in miles per hour) for cars on an expressway as a function of the number of cars on that expressway. Notice from the form of the curve that as the number of cars increases, the safe velocity decreases. As the graph shows, increasing the number of cars on the road to 80, decreases the safe velocity to 38 mph. A further increase in the number of cars on the road to 200, decreases the safe velocity to 20 mph.

Hence, when that radio announcer says, “There is no accident on the road, the heavy traffic comes from volume,” he means that by increasing the number of cars on the road, the safe velocity of each car must decrease.

You might wonder if there is some optimum number of cars that a road can handle safely. We can define the capacity $C$ of a road as the number of cars that pass a particular place per unit time. Stated mathematically, this is

$$C = \frac{N}{t}$$

(H2.4)
Chapter 2  Kinematics In One Dimension

From the definition of velocity, the time for \( N \) cars to pass through a distance \( L \), when moving at the velocity \( v_0 \), is

\[
t = \frac{L}{v_0}
\]

Substituting this into equation H2.4 gives

\[
C = \frac{N}{t} = \frac{N}{L/v_0} = \frac{v_0}{L/N}
\]

Substituting from equation H2.2 for \( L/N \), the capacity of the road is

\[
C = \frac{v_0}{x_c + \frac{v_0 t_R - v_0^2}{2a}} \quad (H2.5)
\]

Using the same values for \( x_c \), \( t_R \), and \( a \) as before, equation H2.5 is plotted in figure 4. The number of cars per hour that the road can hold is on the \( y \)-axis, and the speed of the cars in miles per hour is on the \( x \)-axis. Notice that at a speed of 60 mph, the road can handle 1200 cars per hour. By decreasing the speed of the cars, the number of cars per hour that the road can handle increases. As shown in the figure, if the speed decreases to 40 mph the road can handle about 1600 cars per hour. Notice that the curve peaks at a speed of about 13 mph, allowing about 2300 cars/hour to flow on the expressway. Thus, according to this model, the optimum speed to pass the greatest number of cars per hour is only 13 mph. Hence, even though the road may be called an expressway, if the volume of cars increases significantly, the cars are not going to travel very rapidly. The solution to the problem is to build more lanes to handle the increased volume.

It should also be emphasized that this model is based on safe driving intervals between cars. If an object were to drop from the back of a truck you are following, you would need the safe distance to stop in time to avoid hitting the object. On the other hand, if the car in front of you, also traveling at 55 mph, has to stop, and if both drivers have the same reaction time and both cars decelerate at the same rate, then both cars will need 231 ft to come to a stop. Hence, when both cars
come to a stop they will still be separated by the distance of 231 ft. For this reason, in areas of very heavy traffic, many people do not leave the safe distance between them and the car in front. Instead, they get closer and closer to the car in front of them until they are only separated by the reaction distance \( x_R \). I call this the kamikaze model, for obvious reasons. The kamikaze model allows more cars to travel at a greater velocity than are allowed by the safe stopping distance model. The velocity of the cars as a function of the number of cars is found by solving equation H2.2 with the \( v_0^2 \) term, which is the term associated with the deceleration distance \( x_d \) set equal to zero. The result is shown in figure 5, which compares the

![Figure 5](image)

**Figure 5** Comparison of traffic with the safe stopping distance model and the kamikaze model.

The capacity of the expressway for the kamikaze model is found by setting the \( v_0^2 \) term in equation H2.5 to zero. The result is shown in figure 6. Notice that in the kamikaze model the capacity increases with velocity, and there is no optimum speed for the maximum car flow. In practice, the actual capacity of an expressway lies somewhere between these two extremes.

In conclusion, if your expressway is not much of an expressway, it is time to petition your legislators to allocate more money for the widening of the expressway, or maybe it is time to move to a less populated part of the country.
Figure 6  Comparison of the capacity versus velocity for the safe stopping distance model and the kamikaze model.

The Language of Physics

Kinematics
The branch of mechanics that describes the motion of a body without regard to the cause of that motion (p.).

Average velocity
The average rate at which the displacement vector changes with time. Since a displacement is a vector, the velocity is also a vector (p.).

Average speed
The distance that a body moves per unit time. Speed is a scalar quantity (p.).

Constant velocity
A body moving in one direction in such a way that it always travels equal distances in equal times (p.).

Acceleration
The rate at which the velocity of a moving body changes with time (p.).

Instantaneous velocity
The velocity at a particular instant of time. It is defined as the limit of the ratio of the change in the displacement of the body to the change in time, as the time interval approaches zero. The magnitude of the instantaneous velocity is the instantaneous speed of the moving body (p.).
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**Kinematic equations of linear motion**
A set of equations that gives the displacement and velocity of the moving body at any instant of time, and the velocity of the moving body at any displacement, if the acceleration of the body is a constant (p. ).

**Freely falling body**
Any body that is moving under the influence of gravity only. Hence, any body that is dropped or thrown on the surface of the earth is a freely falling body (p. ).

**Acceleration due to gravity**
If air friction is ignored, all objects that are dropped near the surface of the earth, are accelerated toward the center of the earth with an acceleration of 9.80 m/s$^2$ (p. ).

**Projectile motion**
The motion of a body thrown or fired with an initial velocity $v_0$ in a gravitational field (p. ).

**Trajectory**
The path through space followed by a projectile (p. ).

**Summary of Important Equations**

- **Average velocity**
  \[ v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \]  
  \[ (2.5) \]

- **Average Acceleration**
  \[ a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \]  
  \[ (2.7) \]

- **Instantaneous velocity**
  \[ v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \]  
  \[ (2.8) \]

- **Instantaneous Acceleration**
  \[ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \]  
  \[ (2.9) \]

- **Velocity at any time for constant acceleration**
  \[ v = v_0 + at \]  
  \[ (2.11) \]

- **Displacement at any time for constant acceleration**
  \[ x = v_0 t + \frac{1}{2} at^2 \]  
  \[ (2.14) \]

- **Velocity at any displacement**
  \[ v^2 = v_0^2 + 2ax \]  
  \[ (2.16) \]

- **Velocity of a moving body**
  \[ v - v_0 = \int_{0}^{t} adt \]  
  \[ (2.36) \]

- **Displacement of a moving body**
  \[ x - x_0 = \int_{0}^{t} v_0 dt + \int_{0}^{t} at dt \]
Questions for Chapter 2

1. Discuss the difference between distance and displacement.
2. Discuss the difference between speed and velocity.
3. Discuss the difference between average speed and instantaneous speed.
4. Although speed is the magnitude of the instantaneous velocity, is the average speed equal to the magnitude of the average velocity?
5. Why can the kinematic equations be used only for motion at constant acceleration?
6. In dealing with average velocities discuss the statement, “Straight line motion at 60 km/hr for 1 hr followed by motion in the same direction at 30 km/hr for 2 hr does not give an average of 45 km/hr but rather 40 km/hr.”
7. What effect would air resistance have on the velocity of a body that is dropped near the surface of the earth?
8. What is the acceleration of a projectile when its instantaneous vertical velocity is zero at the top of its trajectory?
9. Can an object have zero velocity at the same time that it has an acceleration? Explain and give some examples.
10. Can the velocity of an object be in a different direction than the acceleration? Give some examples.
11. A person on a moving train throws a ball straight upward. Describe the motion as seen by a person on the train and by a person on the station platform.
12. You are in free fall, and you let go of your watch. What is the relative velocity of the watch with respect to you?
13. What kind of motion is indicated by a graph of displacement versus time, if the slope of the curve is (a) horizontal, (b) sloping upward to the right, and (c) sloping downward?
14. What kind of motion is indicated by a graph of velocity versus time, if the slope of the curve is (a) horizontal, (b) sloping upward at a constant value, (c) sloping upward at a changing rate, (d) sloping downward at a constant value, and (e) sloping downward at a changing rate?

Hints for Problem Solving

To be successful in a physics course it is necessary to be able to solve problems. The following procedure should prove helpful in solving the physics problems assigned. First, as a preliminary step, read the appropriate topic in the textbook. Do not attempt to solve the problems before doing this. Look at the appropriate illustrative problems to see how they are solved. With this background, now read the assigned problem. Now continue with the following procedure.

1. Draw a small picture showing the details of the problem. This is very useful so that you do not lose sight of the problem that you are trying to solve.
2. List all the information that you are given.
3. List all the answers you are expected to find.
4. From the summary of important equations or the text proper, list the equations that are appropriate to this topic.
5. Pick the equation that relates the variables that you are given.
6. Place a check mark (✓) over each variable that is given and a question mark (?) over each variable that you are looking for.
7. Solve the equation for the unknown variable.
8. When the answer is obtained check to see if the answer is reasonable.

Let us apply this technique to the following example.

A car is traveling at 10.0 m/s when it starts to accelerate at 3.00 m/s². Find (a) the velocity and (b) the displacement of the car at the end of 5 s.

1. Draw a picture of the problem.

2. Given: 
   \[ v_0 = 10.0 \text{ m/s} \]
   \[ a = 3.00 \text{ m/s}^2 \]
   \[ t = 5 \text{ s} \]
3. Find: 
   \[ v = ? \]
   \[ x = ? \]

4. The problem is one in kinematics and the kinematic equations apply. That is,
   
   (1) \[ x = v_0 t + \frac{1}{2} at^2 \]
   (2) \[ v = v_0 + at \]
   (3) \[ v^2 = v_0^2 + 2ax \]

5. Part a of the problem. To solve for the velocity \( v \), we need an equation containing \( v \). Equation 1 does not contain a velocity term \( v \), and hence can not be used to solve for the velocity. Equations 2 and 3, on the other hand, both contain \( v \). Thus, we can use one or possibly both of these equations to solve for the velocity.

6. Write down the equation and place a check mark over the known terms and a question mark over the unknown terms:

   \[ ? \quad ✓ \quad ✓ \]
   (2) \[ v = v_0 + at \]
The only unknown in equation 2 is the velocity $v$ and we can now solve for it.

7. The velocity after 5 s, found from equation 2 is

$$v = v_0 + at$$

$$= 10.0 \text{ m/s} + (3.00 \text{ m/s}^2)(5 \text{ s})$$

$$= 10.0 \text{ m/s} + 15.0 \text{ m/s}$$

$$= 25.0 \text{ m/s}$$

Notice what would happen if we tried to use equation 3 at this time:

$$v^2 = v_0^2 + 2ax$$

We cannot solve for the velocity $v$ from equation 3 because there are two unknowns, both $v$ and $x$. However, if we had solved part b of the problem for $x$ first, then we could have used this equation.

5. Part b of the problem. To solve for the displacement $x$, we need an equation containing $x$. Notice that equation 2 does not contain $x$, so we cannot use it. Equations 1 and 3, on the other hand, do contain $x$, and we can use either to solve for $x$.

6. Looking at equation 1, we have

$$x = v_0t + \frac{1}{2}at^2$$

7. Solving for the only unknown in equation 1, $x$, we get

$$x = v_0t + \frac{1}{2}at^2$$

$$= (10.0 \text{ m/s})(5 \text{ s}) + \frac{1}{2}(3.00 \text{ m/s}^2)(5 \text{ s})^2$$

$$= 50 \text{ m} + 37.5 \text{ m}$$

$$= 87.5 \text{ m}$$

Note that at this point we could also have used equation 3 to determine $x$, because we already found the velocity $v$ in part a of the problem.

**Problems for Chapter 2**

2.1 Kinematics - The Study of Motion

1. A driver travels 500 km in 5 hr and 25 min. What is his average speed in (a) km/hr, and (b) m/s?
2. A car travels at 65.0 km/hr for 2 hr and 100 km/hr for 3 hr. What is its average speed?

3. A man hears the sound of thunder 5 s after he sees the lightning flash. If the speed of sound in air is 343 m/s, how far away is the lightning? Assume that the speed of light is so large that the lightning was seen essentially at the same time that it was created.

4. The earth-moon distance is $3.84 \times 10^8$ m. If it takes 3 days to get to the moon, what is the average speed?

5. Electronic transmission is broadcast at the speed of light, which is $3.00 \times 10^8$ m/s. How long would it take for a radio transmission from earth to an astronaut orbiting the planet Mars? Assume that at the time of transmission the distance from earth to Mars is $7.80 \times 10^7$ km.

6. In the game of baseball, some excellent fast-ball pitchers have managed to pitch a ball at approximately 160 km/hr. If the pitcher’s mound is 18.5 m from home plate, how long does it take the ball to get to home plate? If the pitcher then throws a change-of-pace ball (a slow ball) at 95.0 km/hr, how long will it now take the ball to get to the plate?

7. A plot of the displacement of a car as a function of time is shown in the diagram. Find the velocity of the car along the paths (a) $O-A$, (b) $A-B$, (c) $B-C$, and (d) $C-D$.

8. A plot of the velocity of a car as a function of time is shown in the diagram. Find the acceleration of the car along the paths (a) $O-A$, (b) $A-B$, (c) $B-C$, and (d) $C-D$. 

Diagram for problem 7.
2.6 The Kinematic Equations

9. A girl who is initially running at 1.00 m/s increases her velocity to 2.50 m/s in 5.00 s. Find her acceleration.

10. A car is traveling at 95.0 km/hr. The driver steps on the brakes and the car comes to a stop in 60.0 m. What is the car’s deceleration?

11. A train accelerates from an initial velocity of 25.0 km/hr to a final velocity of 65.0 km/hr in 8.50 s. Find its acceleration and the distance the train travels during this time.

12. An airplane travels 450 m at a constant acceleration while taking off. If it starts from rest, and takes off in 15.0 s, what is its takeoff velocity?

13. A car starts from rest and acquires a velocity of 30.0 km/hr in 10.0 s. Where is the car located and what is its velocity at 10.0, 15.0, 20.0, and 25.0 s?

14. A jet airplane goes from rest to a velocity of 75.0 m/s in a distance of 725 m. What is the airplane’s average acceleration in m/s²?

15. An electron in a vacuum tube acquires a velocity of $5.3 \times 10^8$ cm/s in a distance of 0.25 cm. Find the acceleration of the electron.

16. A driver traveling at 100 km/hr tries to stop the car and finds that the brakes have failed. The emergency brake is then pulled and the car comes to a stop in 130 m. Find the car’s deceleration.

17. An airplane has a touchdown speed of 140 km/hr and comes to rest in 120 m. What is the airplane’s average deceleration? How long does it take the plane to stop?
18. A pitcher gives a baseball a horizontal velocity of 30.0 m/s by moving his arm through a distance of approximately 2.50 m. What is the average acceleration of the ball during this throwing process?

19. The speedometer of a car reads 95.0 km/hr when the brakes are applied. The car comes to rest in 4.55 s. How far does the car travel before coming to rest?

20. A body with unknown initial velocity moves with constant acceleration. At the end of 8.00 s, it is moving at a velocity of 50.0 m/s and it is 200 m from where it started. Find the body’s acceleration and its initial velocity.

21. A driver traveling at 30.0 km/hr sees the light turn red at the intersection. If his reaction time is 0.600 s, and the car can decelerate at 4.50 m/s², find the stopping distance of the car. What would the stopping distance be if the car were moving at 90.0 km/hr?

22. A uniformly accelerating train passes a green light signal at 25.0 km/hr. It passes a second light 125 m farther down the track, 12.0 s later. What is the train’s acceleration? What is the train’s velocity at the second light?

![Diagram for problem 22.](image)

23. A car accelerates from 80.0 km/hr to 130 km/hr in 26.9 s. Find its acceleration and the distance the car travels in this time.

24. A motorcycle starts from rest and accelerates at 4.00 m/s² for 5.00 s. It then moves at constant velocity for 25.0 s, and then decelerates at 2.00 m/s² until it stops. Find the total distance that the motorcycle has moved.

25. A car starts from rest and accelerates at a constant rate of 3.00 m/s² until it is moving at 18.0 m/s. The car then decreases its acceleration to 0.500 m/s² and continues moving for an additional distance of 250 m. Find the total time taken.

2.7 The Freely Falling Body

26. A passenger, in abandoning a sinking ship, steps over the side. The deck is 15.0 m above the water surface. With what velocity does the passenger hit the water?

27. How long does it take for a stone to fall from a bridge to the water 30.0 m below? With what velocity does the stone hit the water?

28. An automobile traveling at 95.0 km/hr hits a stone wall. From what height would the car have to fall to acquire the same velocity?

29. A rock is dropped from the top of a building and hits the ground 8.00 s later. How high is the building?

30. A stone is dropped from a bridge 30.0 m high. How long will it take for the stone to hit the water below?
31. A ball is dropped from a building 50.0 meters high. How long will it take the ball to hit the ground below?

32. A girl is standing in an elevator that is moving upward at a velocity of 3.75 m/s when she drops her handbag. If she was originally holding the bag at a height of 1.25 m above the elevator floor, how long will it take the bag to hit the floor?

2.9 Projectile Motion in One Dimension

33. A ball is thrown vertically upward with an initial velocity of 40.0 m/s. Find its position and velocity at the end of 2, 4, 6, and 8 s and sketch these positions and velocities on a piece of graph paper.

34. A projectile is fired vertically upward with an initial velocity of 40.0 m/s. Find the position and velocity of the projectile at 1, 3, 5, and 7 s.

35. A ball is thrown vertically upward from the top of a building 40.0 m high with an initial velocity of 25.0 m/s. What is the total time that the ball is in the air?

36. A stone is thrown vertically upward from a bridge 30.0 m high at an initial velocity of 15.0 m/s. How long will it take for the stone to hit the water below?

37. A stone is thrown vertically downward from a bridge 30.0 m high at an initial velocity of −15.0 m/s. How long will it take for the stone to hit the water below?

38. A rock is thrown vertically downward from a building 40.0 m high at an initial velocity of −15.0 m/s. (a) What is the rock’s velocity as it strikes the ground? (b) How long does it take for the rock to hit the ground?

39. A baseball batter fouls a ball vertically upward. The ball is caught right behind home plate at the same height that it was hit. How long was the baseball in flight if it rose a distance of 30.0 m? What was the initial velocity of the baseball?

2.12 The Calculus and Kinematics

40. A body moves in the \(x\)-direction with an acceleration given by

\[ a = (2.5 \text{ m/s}^3)t + (1.55 \text{ m/s}^3)t^2 \]

Find the equations for the velocity and displacement of the moving body at any time \(t\). The body starts from rest at \(x_0 = 0\). Find the numerical values of the velocity, displacement, and acceleration at \(t = 5.00\) s.

41. A body moves in the \(x\)-direction with an acceleration given by

\[ a = A \cos 2t \]

Find the equations for the velocity and displacement of the moving body at any time \(t\).
Chapter 2  Kinematics In One Dimension

Additional Problems

42. A missile has a velocity of 16,000 km/hr at “burn-out,” which occurs 2 min after ignition. Find the average acceleration in (a) m/s², and (b) in terms of g, the acceleration due to gravity at the surface of the earth.

43. A block slides down a smooth inclined plane that makes an angle of 25.0° with the horizontal. Find the acceleration of the block. If the plane is 10.0 meters long and the block starts from rest, what is its velocity at the bottom of the plane? How long does it take for the block to get to the bottom?

44. At the instant that the traffic light turns green, a car starting from rest with an acceleration of 2.50 m/s² is passed by a truck moving at a constant velocity of 60.0 km/hr. (a) How long will it take for the car to overtake the truck? (b) How far from the starting point will the car overtake the truck? (c) At what velocity will the car be moving when it overtakes the truck?

45. A boat passes a buoy while moving to the right at a velocity of 8.00 m/s. The boat has a constant acceleration to the left, and 10.0 s later the boat is found to be moving at a velocity of −3.00 m/s. Find (a) the acceleration of the boat, (b) the distance from the buoy when the boat reversed direction, (c) the time for the boat to return to the buoy, and (d) the velocity of the boat when it returns to the buoy.

46. Two trains are initially at rest on parallel tracks with train one 50.0 m ahead of train two. Both trains accelerate simultaneously, train 1 at the rate of 2.00 m/s² and train 2 at the rate of 2.50 m/s². How long will it take train 2 to overtake train 1? How far will train 2 travel before it overtakes train 1?

47. Repeat problem 46 but with train 1 initially moving at 5.00 m/s and train 2 initially moving at 7.00 m/s.

48. A policewoman driving at 80.0 km/hr observes a car 50.0 m ahead of her speeding at 120 km/hr. If the county line is 400 m away from the police car, what must the acceleration of the police car be in order to catch the speeder before he leaves the county?

49. Two trains are approaching each other along a straight and level track. The first train is heading south at 125 km/hr, while the second train is heading north at 80.0 km/hr. When they are 2.00 km apart, they see each other and start to decelerate. Train 1 decelerates at 2.00 m/s², while train 2 decelerates at 1.50 m/s². Will the trains be able to stop or will there be a collision?

50. A boy in an elevator, which is descending at the constant velocity of −5.00 m/s, jumps to a height of 0.500 m above the elevator floor. How far will the elevator descend before the boy returns to the elevator floor?

51. The acceleration due to gravity on the moon is 1.62 m/s². If an astronaut on the moon throws a ball straight upward, with an initial velocity of 25.0 m/s, how high will the ball rise?

52. A helicopter, at an altitude of 300 m, is rising vertically at 20.0 m/s when a wheel falls off. How high will the wheel go with respect to the ground? How long will it take for the wheel to hit the ground below? At what velocity will the wheel hit the ground?
53. A ball is dropped from the roof of a building 40.0 m high. Simultaneously, another ball is thrown upward from the ground and collides with the first ball at half the distance to the roof. What was the initial velocity of the ball that was thrown upward?

54. A ball is dropped from the top of a 40.0-m high building. At what initial velocity must a second ball be thrown from the top of the building 2.00 s later, such that both balls arrive at the ground at the same time?

**Interactive Tutorials**

55. A train accelerates from an initial velocity of 20.0 m/s to a final velocity of 35.0 m/s in 11.8 s. Find its acceleration and the distance the train travels in this time.

56. A ball is dropped from a building 50.0 m high. How long will it take the ball to hit the ground below and with what final velocity?

57. Instantaneous velocity. If the equation for the displacement $x$ of a body is known, the average velocity throughout an interval can be computed by the formula

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

The instantaneous velocity is defined as the limit of the average velocity as $\Delta t$ approaches zero. That is,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

For an acceleration with a displacement given by $x = 0.5 \, at^2$, use different values of $\Delta t$ to see how the average velocity approaches the instantaneous velocity. Compare this to the velocity determined by the equation $v = at$, and determine the percentage error. Plot the average velocity, $(\Delta x)/(\Delta t)$, versus $\Delta t$.

58. Free-fall and generalized one-dimensional projectile motion. A projectile is fired from a height $y_0$ above the ground with an initial velocity $v_0$ in a vertical direction. Find (a) the time $t_r$ for the projectile to rise to its maximum height, (b) the
total time $t_t$ the ball is in the air, (c) the maximum height $y_{\text{max}}$ of the projectile, (d) the velocity $v_g$ of the projectile as it strikes the ground, and (e) the location and velocity of the projectile at any time $t$. (f) Plot a picture of the motion as a function of time.

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