Chapter 22: Electric Flux and Gauss’s Law

22.1 Introduction
We have seen in chapter 21 that determining the electric field of a continuous charge distribution can become very complicated for some charge distributions. It would be desirable if we could find a simpler way to determine the electric field of a charge distribution. It turns out that if a certain symmetry exists in the charge distribution it is possible to determine the electric field by means of Gauss’s law. To understand Gauss’s law we must first understand the concept of electric flux.

22.2 Electric Flux
Flux is a quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface. Figure 22.1a, shows an electric field $E$ passing through a portion of a surface of area $A$. The area of the surface is represented by a vector $A$, whose magnitude is the area $A$ of the surface, and whose direction is perpendicular to the surface. That an area can be represented by a vector was shown in Chapter 3. The electric flux is defined to be

$$\Phi_E = E \cdot A = EA \cos \theta$$  

(22.1)

and is a quantitative measure of the number of lines of $E$ that pass normally through the surface area $A$. The number of lines represents the strength of the field. The vector $E$, at the point $P$ of figure 22.1(a), can be resolved into the components, $E_\perp$ the component perpendicular to the surface, and $E_\parallel$ the parallel component. The perpendicular component is given by

$$E_\perp = E \cos \theta$$  

(22.2)

while the parallel component is given by

$$E_\parallel = E \sin \theta$$  

(22.3)

The parallel component $E_\parallel$ lies in the surface itself and therefore does not pass through the surface, while the perpendicular component $E_\perp$ completely passes through the surface at the point $P$. The product of the perpendicular component $E_\perp$ and the area $A$

$$E_{\perp A} = (E \cos \theta)A = EA \cos \theta = E \cdot A = \Phi_E$$  

(22.4)

is therefore a quantitative measure of the number of lines of $E$ passing normally through the entire surface area $A$. If the angle $\theta$ in equation 22.1 is zero, then $E$ is parallel to the vector $A$ and all the lines of $E$ pass normally through the surface.
area $A$, as seen in figure 22.1(b). If the angle $\theta$ in equation 22.1 is $90^\circ$ then $\mathbf{E}$ is perpendicular to the area vector $\mathbf{A}$, and none of the lines of $\mathbf{E}$ pass through the surface $A$ as seen in figure 22.1(c). The concept of flux is a very important one, and one that will be used frequently later.

**Example 22.1**

*Electric flux.* An electric field of 500 V/m makes an angle of $30.0^\circ$ with the surface vector, which has a magnitude of 0.500 m$^2$. Find the electric flux that passes through the surface.

**Solution**

The electric flux passing through the surface is given by equation 22.1 as
\[ \Phi_E = \mathbf{E} \cdot \mathbf{A} = E A \cos \theta \]
\[ \Phi_E = (500 \text{ V/m})(0.500 \text{ m}^2)\cos 30.0^\circ \]
\[ \Phi_E = 217 \text{ V m} \]

Notice that the unit of electric flux is a volt times a meter.

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### 22.3 Gauss’s Law for Electricity

It was pointed out in section 22.2 that an electric flux was a quantitative measure of the number of electric field lines passing normally through an area. The electric flux was illustrated in figure 22.1 and defined in equation 22.1 as

\[ \Phi_E = \mathbf{E} \cdot \mathbf{A} = E A \cos \theta \quad (22.1) \]

Let us now consider the amount of electric flux that emanates from a positive point charge. Figure 22.2 shows a positive point charge surrounded by an imaginary spherical surface called a Gaussian surface. Let us measure the amount of electric flux through the sphere. The direction of the \( \mathbf{E} \) field is different at every point, however, so equation 22.1 can not be used in its present form. Instead, the spherical surface is broken up into a large number of infinitesimal surface areas \( d\mathbf{A} \), figure 22.2(b), and the infinitesimal amount of flux \( d\Phi_E \) through each of these little areas is computed, i.e.,

**Figure 22.2** Gauss’s law for electricity.
The total flux out of the Gaussian surface becomes the sum or integral of all the
infinitesimal fluxes \( d\Phi_E \), through all the infinitesimal areas \( dA \), that is

\[
\Phi_E = \oint d\Phi_E = \oint E \cdot dA
\]  

(22.6)

The integral symbol \( \oint \) means that the integration is performed over the entire
closed surface that the flux is passing through. The electric vector \( \mathbf{E} \) is everywhere
radial from the point charge \( q \), and \( dA \) is also everywhere radial, hence

\[
\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E \, dA \cos \theta
\]  

(22.7)

Thus,

\[
\Phi_E = \oint E \, dA
\]  

(22.8)

But it was found in chapter 21 that the electric field of a point charge was given by
equation 21.2 as

\[
E = \frac{kq}{r^2}
\]  

(21.2)

Substituting the electric field of a point charge, from equation 21.2, into equation
22.8, we get for the electric flux through the spherical surface

\[
\Phi_E = \oint k \frac{q}{r^2} \, dA
\]  

(22.9)

where \( r \) is the radius of the spherical Gaussian surface, and for that sphere it is a
constant. The terms \( k \) and \( q \) are also constants and they can, therefore, be taken
outside of the integral sign. Thus,

\[
\Phi_E = k \frac{q}{r^2} \oint dA
\]  

(22.10)

But the integral of all the elements of area \( dA \) is equal to the entire surface area of
the sphere. Since the surface area of a sphere is \( 4\pi r^2 \), we have

\[
\oint dA = 4\pi r^2
\]

Thus, the electric flux emanating from a point charge becomes

\[
\Phi_E = k \frac{q}{r^2} 4\pi r^2
\]  

(22.11)

\[
\Phi_E = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} 4\pi r^2
\]  

(22.12)

where use has been made of the fact that \( k \) in equation 22.11 is equal to \( 1/(4\pi \varepsilon_0) \).
Hence, the electric flux associated with a point charge is
Equation 22.13 is Gauss’s law for electricity, and it says that the electric flux $\Phi_E$ that passes through a surface surrounding the point charge $q$ is a measure of the amount of charge $q$ contained within the Gaussian surface. Although equation 22.13 was derived for a point charge, it is true, in general, for any kind of charge distribution. Since $\Phi_E$ was initially defined in equation 22.6, it can be combined with equation 22.13 into the generalization of Gauss’s law as

$$\Phi_E = \oint E \cdot dA = \frac{q}{\epsilon_0}$$

(22.14)

Where $q$ is now the net charge contained within the Gaussian surface. Equation 22.14 was derived on the basis of a point charge. If the charge is distributed over a volume $V$, with a volume charge density $\rho$, then the charge $q$ can be written as

$$q = \int \rho \, dV$$

(22.15)

and Gauss’s law can also be written as

$$\Phi_E = \oint E \cdot dA = \frac{1}{\epsilon_0} \int \rho \, dV$$

(22.16)

If the total charge $q$ within the Gaussian surface is known, Gauss’s law in the form of equation 22.14 can be used. When only the charge density $\rho$ is known, then Gauss’s law in the form of equation 22.16 is used.

When $\Phi_E$ is a positive quantity, the Gaussian surface surrounds a source of positive charge, and electric flux diverges out of the surface. If the point charge is negative then the electric field would go inward, through the Gaussian surface, to the point charge. The vector $E$ would, therefore, make an angle of $180^\circ$ with the area vectors $dA$ and their dot product would be

$$E \cdot dA = E \, dA \cos 180^\circ = -E \, dA$$

Thus, the flux passing through the Gaussian surface would be negative. Hence, whenever $\Phi_E$ is negative, the Gaussian surface surrounds a negative charge distribution and the electric flux converges into the Gaussian surface. If there is no enclosed charge, $q = 0$, and hence, $\Phi_E = 0$. In this case, whatever electric flux enters one part of a Gaussian surface, the same amount must leave somewhere else. These different possibilities are shown for an electric dipole in figure 22.3. Thus, Gaussian surface 1 shows the electric flux diverging from the positive point charge. Gaussian surface 2 shows the electric flux converging into the negative point charge, and Gaussian surface 3 shows no enclosed charge, and the amount of electric field
entering normally through the surface is equal to the amount leaving the surface normally.

Let us now consider some examples of the application of Gauss's law to determine the electric field of various symmetric charge distributions.

### 22.4 The Electric Field of a Spherically Symmetric Uniform Charge Distribution

Figure 22.4(a) shows a spherically symmetric distribution of total electric charge $q$. We assume that the distribution is uniform over the sphere of radius $R$ and has a volume charge density $\rho$ \((\text{C/m}^3)\).

![Figure 22.3 Gaussian surfaces and an electric dipole.](image)

Figure 22.4  A spherically symmetric distribution of electric charge.

Let us find the electric field $\mathbf{E}$ at (a) points outside the sphere, that is for $r > R$ and (b) for points inside the sphere, that is for $r < R$. 

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(a) The electric field outside the uniform charge distribution.

To determine the electric field outside the spherical distribution of charge we first must draw a Gaussian surface. The shape of the Gaussian surface depends on the symmetry of the problem. Since the inherent symmetry in this problem is spherical, we draw a spherical Gaussian surface of radius $r$ around the spherical charge distribution as shown in figure 22.4b. Gauss’s law, equation 22.14, is applied to this spherical surface as

$$\Phi_E = \oint E \cdot dA = \frac{q}{\varepsilon_0}$$  \hspace{1cm} (22.14)$$

From the symmetry of the problem, the electric field $E$ must be radially outward from the charge distribution. (Since the charge distribution is spherical, the most probable direction of the electric field is also spherical, that is, there is no reason to assume that the electric field is more likely to point in one direction than another, that is, there is no reason to assume that the electric field should point in the $x$-direction rather than in some other direction.) The element of area $dA$ of the spherical Gaussian surface is perpendicular to the spherical surface and points outward in the radial direction. Hence the angle between the electric field vector $E$ and the element of area vector $dA$ is $0^\circ$. Gauss’s law therefore becomes

$$\oint E \cdot dA = \oint E \, dA \cos 0^\circ = \frac{q}{\varepsilon_0}$$

But again, from the symmetry of the problem, the magnitude of the electric field will be a constant value at the location $r$ of the Gaussian surface. (Again, there is no reason to assume that the magnitude of the electric field is greater in the $x$-direction than in the $y$-direction, or any other direction. Hence, because of the spherical nature of the charge distribution the magnitude of the electric field will be constant anywhere on the Gaussian surface.) Therefore, the electric field $E$ can be taken outside of the integral to yield

$$E \oint dA = \frac{q}{\varepsilon_0}$$

But $\oint dA$ is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and is equal to the total surface area of the Gaussian sphere. Since the surface area of a sphere is $4\pi r^2$ we have

$$\oint dA = 4\pi r^2$$  \hspace{1cm} (22.18)$$

Therefore Gauss’s law becomes

$$E \oint dA = E 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi r^2} \frac{q}{\varepsilon_0} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2}$$

But from Coulomb’s law

$$\frac{1}{4\pi \varepsilon_0} = k$$

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Hence, the electric field $E$ outside the spherical charge distribution is found to be

$$E = k \frac{q}{r^2}$$  \hspace{0.5cm} (22.19)

Thus outside a spherical charge distribution $q$, the electric field looks like the electric field of a point charge.

(b) The electric field inside the uniform charge distribution.

To find the electric field inside the spherical charge distribution, we draw a new spherical Gaussian surface of radius $r$ that encloses an amount of charge $q'$ as shown in figure 22.4c. The amount of flux passing through this Gaussian surface is found by modifying Gauss’s law, equation 22.14, as

$$\Phi_E = \oint E \cdot dA = \frac{q'}{\varepsilon_0}$$  \hspace{0.5cm} (22.20)

where $q'$ is the amount of charge now contained within the new Gaussian surface and is less than the total spherical charge $q$. If the charge distribution is uniform, then the volume charge density $\rho$ is the same for the total charge as it is for the charge enclosed in the new Gaussian surface. That is,

$$\rho = q \frac{1}{V} = q' \frac{1}{V'}$$  \hspace{0.5cm} (22.21)

where $q$ is the total charge enclosed in the total volume $V$ of charge, while $q'$ is the charge enclosed in the volume $V'$ of charge. From equation 22.21 the amount of charge enclosed within the new Gaussian surface can be written as

$$q' = V' q$$  \hspace{0.5cm} (22.22)

But the volume of the total spherical charge is

$$V = \frac{4}{3} \pi R^3$$

and the volume of the charge in the new Gaussian surface is given by

$$V' = \frac{4}{3} \pi r^3$$

Replacing these volumes in equation 22.22 gives

$$q' = \frac{V'}{V} q = \frac{\frac{4}{3} \pi r^3}{\frac{4}{3} \pi R^3} q$$
and the charge \( q' \) contained within the Gaussian surface is

\[
q' = \frac{r^3}{R^3} q
\]  

(22.23)

Replacing the enclosed charge \( q' \), equation 22.23, into Gauss’s law, equation 22.20 we obtain

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \oint \frac{q'}{\varepsilon_0} = \frac{r^3}{\varepsilon_0 R^3} q
\]

(22.24)

Again, from the symmetry of the problem, the electric field \( \mathbf{E} \) must be radially outward from the charge distribution. The element of area \( d\mathbf{A} \) of the spherical Gaussian surface is perpendicular to the spherical surface and also points outward in the radial direction. Hence the angle between the electric field vector \( \mathbf{E} \) and the element of area vector \( d\mathbf{A} \) is 0°. Gauss’s law therefore becomes

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \oint E \, dA \cos 0^\circ = \frac{r^3}{\varepsilon_0 R^3} q
\]

\[
\oint E \, dA = \frac{r^3}{\varepsilon_0 R^3} q
\]

(22.25)

But again, from the symmetry of the problem, the magnitude of the electric field will be a constant value at the location \( r \) of the Gaussian surface. Therefore, the electric field \( E \) can be taken outside of the integral to yield

\[
E \oint dA = \frac{r^3}{\varepsilon_0 R^3} q
\]

But \( \oint dA \) is the sum of all the infinitesimal elements of area distributed over the entire surface of the Gaussian sphere and hence is equal to the total surface area of the Gaussian sphere. Since the surface area of a sphere is \( 4\pi r^2 \) we have

\[
\oint dA = 4\pi r^2
\]

Therefore Gauss’s law becomes

\[
E \oint dA = E(4\pi r^2) = \frac{r^3}{\varepsilon_0 R^3} q
\]

\[
E = \frac{1}{4\pi r^2} \frac{r^3}{\varepsilon_0 R^3} q = \frac{1}{4\pi \varepsilon_0} \frac{r}{R^3} q
\]

But from Coulomb’s law

\[
\frac{1}{4\pi \varepsilon_0} = k
\]

Hence, the electric field \( E \) inside the spherical charge distribution is found to be

\[
E = k \frac{r}{R^3} q
\]

(22.26)
Equation 22.26 says that the electric field intensity $E$ inside the charge distribution is directly proportional to the radial distance $r$ from the center of the charge distribution, whereas the electric field $E$ outside the charge distribution, equation 22.19, is inversely proportional to the square of the radial distance $r$ from the center of the charge distribution.

Note that when $r$, the radius of the Gaussian surface, is equal to $R$, the radius of the charge distribution, in equation 22.26 (for the electric field intensity $E$ inside the charge distribution), the electric field intensity $E$ at the surface of the charge distribution becomes

$$E = k \frac{r}{R^3} q = k \frac{R}{R^3} q = k \frac{q}{R^2}$$

Also note that when $r$ the radius of the Gaussian surface is equal to $R$ the radius of the charge distribution in equation 22.19 (the electric field intensity $E$ outside the charge distribution), the electric field intensity $E$ at the surface of the charge distribution becomes

$$E = \frac{q}{R^2}$$

which shows that the two results agree at the edge of the charge distribution as they must.

A plot of the electric field intensity $E$ as a function of the radial distance $r$ is shown in figure 22.4(d). It shows the linear relation between $E$ and $r$ within the charge distribution and the inverse square relationship outside the charge distribution.

**Example 22.2**

Electric field of a spherical charge distribution. A spherical charge distribution of $7.52 \times 10^{-5}$ C has a radius of $5.25 \times 10^{-3}$ m. Find the electric field intensity $E$ at (a) 5.00 cm from the charge and (b) $3.25 \times 10^{-3}$ m inside the charge.

**Solution**

a. The electric field intensity $E$ outside a uniform charge distribution is found from equation 22.19 as

$$E = k \frac{q}{r^2}$$

$$E = (9.00 \times 10^9 \text{ N m}^2/\text{C}^2) \frac{(7.52 \times 10^{-5} \text{ C})}{(0.050 \text{ m})^2}$$

$$E = 2.71 \times 10^8 \text{ N/C}$$

b. The electric field intensity $E$ inside a uniform charge distribution is found from equation 22.26 as

$$E = k \frac{r}{R^3} q$$

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22.5 The Electric Field of an infinite line of Charge

Using Gauss’s law, let us determine the electric field intensity $E$ at a distance $r$ from an infinite line of charge lying on the $x$-axis. A small portion of the infinite line of charge is shown in figure 22.5a. We assume that the charge is uniformly distributed and has a linear charge density $\lambda$. (Recall that the linear charge density $\lambda$ is the charge per unit length, that is, $\lambda = q/L$.) To determine the electric field we must first draw a Gaussian surface. In the previous examples of Gauss’s law we used a spherical Gaussian surface because of the spherical symmetry of the problem. However, the line of charge does not have any spherical symmetry and a spherical Gaussian surface can not be used. The line of charge does have a cylindrical symmetry however. Therefore, we draw a cylindrical Gaussian surface of radius $r$ and length $L$ around the infinite line of charge as shown in figure 22.5b. Gauss’s law for the total flux emerging from the Gaussian cylinder, equation 22.14, is applied.

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon_0}$$

The integral in Gauss’s law is over the entire Gaussian surface. We can break the entire cylindrical Gaussian surface into three surfaces. Surface I is the end cap on
the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 22.5b. The total flux $\Phi$ through the entire Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_1 + \Phi_{II} + \Phi_{III}$$  \hspace{1cm} (22.27)

where

- $\Phi_1$ is the electric flux through surface I
- $\Phi_{II}$ is the electric flux through surface II
- $\Phi_{III}$ is the electric flux through surface III

Hence, Gauss’s law becomes

$$\oint E \cdot dA = \oint E \cdot dA + \oint E \cdot dA + \oint E \cdot dA = \frac{q}{\varepsilon_0}$$  \hspace{1cm} (22.28)

Notice that the normal integral sign $\oint$ is now used instead of $\int$, because each integration is now over only a portion of the entire closed surface. Along cylindrical surface I, $dA$ is everywhere perpendicular to the surface and points toward the left as shown in figure 22.5b. The electric field intensity vector $E$ lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector $dA$ of surface I and hence $\theta = 90^0$. Therefore the electric flux through surface I is

$$\Phi_1 = \oint I E \cdot dA = \oint I E dA \cos \theta = \oint I E dA \cos 90^0 = 0$$

Surface II is the cylindrical surface itself. As can be seen in figure 22.5(b), $E$ is everywhere perpendicular to the cylindrical surface pointing outward, and the area vector $dA$ is also perpendicular to the surface and also points outward. Hence $E$ and $dA$ are parallel to each other and the angle $\theta$ between $E$ and $dA$ is zero. Hence, the flux through surface II is

$$\Phi_{II} = \oint I E \cdot dA = \oint I E dA \cos \theta = \oint I E dA \cos 0^0 = \oint I E dA$$

Along cylindrical surface III, $dA$ is everywhere perpendicular to the surface and points toward the right as shown in figure 22.5(b). The electric field intensity vector $E$ lies in the plane of the end cylinder cap and is everywhere perpendicular to the surface vector $dA$ of surface III and therefore $\theta = 90^0$. Hence the electric flux through surface III is

$$\Phi_{III} = \oint III E \cdot dA = \oint III E dA \cos \theta = \oint III E dA \cos 90^0 = 0$$
Combining the flux through each portion of the cylindrical surfaces, equation 22.28 becomes

\[
\Phi_E = \oint I \cdot dA + \oint II \cdot dA + \oint III \cdot dA = \frac{q}{\varepsilon_0}
\]

\[
\Phi_E = 0 + \oint II \cdot dA + 0 = \frac{q}{\varepsilon_0}
\]

From the symmetry of the problem, the magnitude of the electric field intensity \(E\) is a constant for a fixed distance \(r\) from the line of charge, that is, there is no reason to assume that \(E\) in the \(z\)-direction is any different than \(E\) in the \(y\)-direction. Hence, \(E\) can be taken outside the integral sign to yield

\[
\oint II \cdot dA = E \oint II \cdot dA = \frac{q}{\varepsilon_0}
\]

The integral \(\int dA\) represents the sum of all the elements of area \(dA\), and that sum is just equal to the total area of the cylindrical surface. It is easier to see the total area if we unfold the cylindrical surface as shown in figure 22.5c. One length of the surface is \(L\), the length of the cylinder, while the other length is the unfolded circumference \(2\pi r\) of the end of the cylindrical surface.\(^1\) The total area \(A\) is just the product of the length times the width of the rectangle formed by unfolding the cylindrical surface, that is, \(A = (L)(2\pi r)\). Hence, the integral of \(dA\) is

\[
\int dA = A = (L)(2\pi r)
\]

Thus, Gauss’s law becomes

\[
E \oint II \cdot dA = E(L)(2\pi r) = \frac{q}{\varepsilon_0}
\]

Solving for the magnitude of the electric field \(E\) intensity we get

\[
E = \frac{q}{2\pi\varepsilon_0 rL}
\]

But \(q/L = \lambda\) the linear charge density. Therefore

\[
E = \frac{\lambda}{2\pi\varepsilon_0 r}
\]

Also, since \(k = 1/(4\pi\varepsilon_0)\) then \(2k = 1/(2\pi\varepsilon_0)\). Hence the magnitude of the electric field intensity \(E\) for an infinite line of charge is

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\(^1\) You can try this by taking a sheet of paper and rolling it into a cylinder. The area of that cylinder is found by unrolling the piece of paper into its normal rectangular shape. The area of the rectangle is the product of its length times its width. The length is just the length of the page. The width was the circular portion of the cylinder. But the circumference of a circle is \(2\pi r\). So the unrolled width of the paper is equal to the circumference of the circle of the rolled paper.
Notice that this is the same result we obtained in the last chapter by a direct addition of the electric fields of all the elements of charge. The solution by Gauss's law, however, is much simpler.

**Example 22.3**

*Electric field of an infinite line of charge.* An infinite line of charge carries a linear charge density of \(2.55 \times 10^{-5} \text{ C/m}\). Find the electric field \(E\) at a distance of 15.5 cm from the line of charge.

**Solution**

The electric field intensity \(E\) for an infinite line of charge is found by equation 22.29 as

\[
E = \frac{2k\lambda}{r}
\]

\[
E = \frac{2(9.00 \times 10^9 \text{ Nm}^2/\text{C}^2)(2.55 \times 10^{-5} \text{ C/m})}{0.155 \text{ m}}
\]

\[
E = 2.96 \times 10^6 \text{ N/C}
\]

To go to this Interactive Example click on this sentence.

### 22.6 The Electric Field of an Infinite Plane Sheet of Charge

Using Gauss’s law, let us determine the electric field intensity \(\mathbf{E}\) at a distance \(r\) in front of an infinite plane sheet of charge as shown in figure 22.6(a). We assume that the charge is uniformly distributed over the sheet and has a surface charge density \(\sigma\). (Recall that the surface charge density \(\sigma\) is the charge per unit area, that is, \(\sigma = \frac{q}{A}\).) To determine the electric field we must first draw a Gaussian surface. The type of Gaussian surface drawn depends on the symmetry of the problem. We draw a cylindrical Gaussian surface through the sheet of charge as shown in figure 22.6(b). Gauss’s law for the total flux emerging from the Gaussian cylinder, equation 22.14, is

\[
\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}
\]

The integral in equation 22.14 is over the entire Gaussian surface. As before, we can break the entire surface of the cylinder into three surfaces. Surface I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical
surface itself, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 22.6(b). The total flux $\Phi_E$ through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III}$$

where

$\Phi_I$ is the electric flux through surface I
$\Phi_{II}$ is the electric flux through surface II
$\Phi_{III}$ is the electric flux through surface III

Gauss’s law becomes

$$\Phi_E = \oint E \cdot dA = \int_E E \cdot dA + \int_{II} E \cdot dA + \int_{III} E \cdot dA = \frac{q}{\varepsilon_0} \quad (22.30)$$

Surface I is the end cap on the left-hand side of the cylinder and as can be seen from figure 22.6b, the electric field vector $E$ points toward the left and since the area vector $dA$ is perpendicular to the surface pointing outward it also points to the left. Hence $E$ and $dA$ are parallel to each other and the angle $\theta$ between $E$ and $dA$ is zero. Hence, the flux through surface I is

$$\Phi_I = \int_I E \cdot dA = \int_I E \, dA \cos \theta = \int_I E \, dA \cos 0^\circ = \int_I E \, dA$$

Along cylindrical surface II, $dA$ is everywhere perpendicular to the surface as shown in figure 22.6(b). $E$ lies in the cylindrical Gaussian surface and is everywhere perpendicular to the surface vector $dA$ on surface II and therefore $\theta = 90^\circ$. Hence the electric flux through surface II is

$$\Phi_{II} = \int_{II} E \cdot dA = \int_{II} E \, dA \cos 90^\circ = 0$$
Surface III is the end cap on the right-hand side of the cylinder and as can be seen from figure 22.6b the electric field vector $E$ points toward the right and since the area vector $dA$ is perpendicular to the surface pointing outward it also points to the right. Hence $E$ and $dA$ are parallel to each other and the angle $\theta$ between $E$ and $dA$ is zero. Hence, the flux through surface III is

$$\Phi_{III} = \int_E dA = \int_E E \cos \theta = \int_E E \cos 90^\circ = 0$$

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, Gauss’s law becomes

$$\Phi_E = \Phi_I + \Phi_{II} + \Phi_{III}$$

$$\Phi_E = \int_E E \, dA + 0 + \int_{III} EdA = \frac{q}{\varepsilon_0}$$

But $E$ has the same magnitude in integral I and III and hence the total flux can be written as

$$\Phi_E = 2 \int_I E \, dA = \frac{q}{\varepsilon_0}$$

But the magnitude of $E$ is a constant in the integral and can be factored out of the integral to yield

$$2E \int \, dA = \frac{q}{\varepsilon_0}$$

But $\int dA = A$ the area of the end cap of the Gaussian cylinder, hence

$$2EA = \frac{q}{\varepsilon_0}$$

$A$ is the magnitude of the area of the Gaussian end cap surface and $q$ is the charge enclosed within that Gaussian surface. Solving for the electric field intensity

$$E = \frac{q}{2\varepsilon_0A} \quad (22.31)$$

But the surface charge density $\sigma$ is defined as the charge per unit area, i.e.,

$$\sigma = \frac{q}{A} \quad (22.32)$$

Combining equation 22.32 with 22.31, gives for the electric field intensity

$$22-16$$
Thus, the electric field in front of an infinite sheet of charge is given by equation 22.33 in terms of the surface charge density and the permittivity of free space. Notice that the electric field is a constant depending only upon the surface charge density \( \sigma \) and not on the distance from the sheet of charge. Also note that in many practical problems with a finite sheet of charge, we can use the result of the infinite sheet of charge as a good approximation if the electric field is found very close to the finite sheet of charge. At a very close distance, the finite sheet of charge can look like an infinite sheet of charge.

\[
E = \frac{\sigma}{2\varepsilon_0} \tag{22.33}
\]

**Example 22.4**

Electric field of an infinite sheet of charge. Find the electric field in front of an infinite sheet of charge carrying a surface charge density of \( 3.58 \times 10^{-12} \text{ C/m}^2 \).

**Solution**

The electric field in front of an infinite sheet of charge is found from equation 22.33 as

\[
E = \frac{3.58 \times 10^{-12} \text{C/m}^2}{2(8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2)} = 0.202 \text{ N/C}
\]

To go to this Interactive Example click on this sentence.

**22.7 The Electric Field Inside a Conducting Body**

Let us take a spherical conductor, such as a solid aluminum ball of radius \( R \). A positively charged rod is touched to the sphere giving it a positive charge \( q \). Let us find the electric field intensity \( E \) inside the spherical conductor. When the rod is touched to the sphere, positive charge is distributed over the sphere. But since like charges repel each other there will be a force of repulsion on each of these charges tending to push them away from each other. Since the body is a conducting body, the charges are free to move and because of the repulsion, move as far away from each other as possible. However, the maximum distance that they can move from each other is just to the surface of the sphere. Hence, the charge \( q \) becomes distributed over the surface of the sphere. That is, there is no charge within the sphere itself. To determine \( E \) inside the conducting sphere, we draw a spherical
Gaussian surface of radius $r$ within the conducting sphere as shown in figure 22.7. Applying Gauss's law to determine the flux through the Gaussian sphere we get

$$\Phi_E = \oint E \cdot dA = \frac{q}{\varepsilon_0}$$

But since we have just shown that $q = 0$ inside the sphere, Gauss's law becomes

$$\Phi_E = \oint E \cdot dA = \frac{q}{\varepsilon_0} = 0$$

$$\oint E \cdot dA = 0$$ \hspace{1cm} (22.34)

The only way that this can be true in general is for

$$E = 0$$ \hspace{1cm} (22.35)

*Hence the electric field within a conductor is always equal to zero.*

*Figure 22.7* The electric field within a conducting body.

### 22.8 The Electric Field Between Two Oppositely Charged Parallel Conducting Plates

Let us determine the electric field between the two oppositely charged conducting plates, shown in figure 22.8(a), by Gauss's law. We start by drawing a Gaussian surface. For the symmetry of this problem we pick a cylinder for the Gaussian surface as shown in the diagram. One end of the Gaussian surface lies within one of the parallel conducting plates while the other end lies in the region between the two conducting plates where we wish to find the electric field. Gauss's law is given by equation 22.14 as

$$\Phi_E = \oint E \cdot dA = \frac{q}{\varepsilon_0}$$ \hspace{1cm} (22.14)

The sum in equation 22.14 is over the entire Gaussian surface. As in the previous sections, we can break the entire surface of the cylinder into three surfaces. Surface
I is the end cap on the left-hand side of the cylinder, surface II is the main cylindrical surface, and surface III is the end cap on the right-hand side of the cylinder as shown in figure 22.8(b). The total flux $\Phi$ through the Gaussian surface is the sum of the flux through each individual surface. That is,

$$\Phi = \Phi_I + \Phi_{II} + \Phi_{III}$$

where

- $\Phi_I$ is the electric flux through surface I
- $\Phi_{II}$ is the electric flux through surface II
- $\Phi_{III}$ is the electric flux through surface III

Gauss’s law becomes

$$\Phi_E = \oint E \cdot dA = \int_I E \cdot dA + \int_{II} E \cdot dA + \int_{III} E \cdot dA = \frac{q}{\varepsilon_0}$$

Because the plate is a conducting body, all charge must reside on its outer surface, hence $E = 0$, inside the conducting body. Since Gaussian surface I lies within the conducting body the electric field on surface I is zero. Hence the flux through surface I is,

$$\Phi_I = \int_I E \cdot dA = \int_I E \, dA \cos \theta = \int_I (0) \, dA \cos \theta = 0$$

Along cylindrical surface II, $dA$ is everywhere perpendicular to the surface as shown in figure 22.8(b). $E$ lies in the cylindrical surface, pointing toward the right, and is everywhere perpendicular to the surface vector $dA$ on surface II, and therefore $\theta = 90^\circ$. Hence the electric flux through surface II is
Surface III is the end cap on the right-hand side of the cylinder and as can be seen from figure 22.8b \( \mathbf{E} \) points toward the right and since the area vector \( d\mathbf{A} \) is perpendicular to the surface pointing outward it also points to the right. Hence \( \mathbf{E} \) and \( d\mathbf{A} \) are parallel to each other and the angle \( \theta \) between \( \mathbf{E} \) and \( d\mathbf{A} \) is zero. Hence, the flux through surface III is

\[
\Phi_{III} = \int_{III} \mathbf{E} \cdot d\mathbf{A} = \int_{III} E \ dA \cos \theta = \int_{III} E \ dA \cos 90^\circ = 0
\]

The total flux through the Gaussian surface is equal to the sum of the fluxes through the individual surfaces. Hence, Gauss’s law becomes

\[
\Phi = \Phi_I + \Phi_{II} + \Phi_{III}
\]

\[
\Phi_E = 0 + 0 + \int_{III} E dA = \frac{q}{\varepsilon_o}
\]

But \( E \) is constant in every term of the sum in integral III and can be factored out of the integral giving

\[
\Phi_E = E \int dA = \frac{q}{\varepsilon_o}
\]

But \( \int dA = A \) the area of the end cap, hence

\[
EA = \frac{q}{\varepsilon_o}
\]

\( A \) is the magnitude of the area of Gaussian surface III and \( q \) is the charge enclosed within Gaussian surface III. Solving for the electric field \( E \) between the conducting plates gives

\[
E = \frac{q}{\varepsilon_o A} \quad (22.36)
\]

Equation 22.36 describes the electric field in terms of the charge on Gaussian surface III and the area of Gaussian surface III. If the surface charge density on the plates is uniform, the surface charge density of Gaussian surface III is the same as the surface charge density of the entire plate. Hence the ratio of \( q/A \) for the Gaussian surface is the same as the ratio \( q/A \) for the entire plate. Thus \( q \) in equation 22.36 can also be interpreted as the total charge \( q \) on the plates, and \( A \) can be interpreted as the total area of the conducting plate. Therefore, equation 22.36 gives the electric field \( E \) between the conducting plates in terms of the charge \( q \) on the plates, the area \( A \) of the plates, and the permittivity \( \varepsilon_o \) of the medium between the plates.

Instead of describing the electric field in terms of the charge \( q \) on the plates and the area \( A \) of the plates, it is sometimes convenient to express this result in
terms of the surface charge density $\sigma$. Since the surface charge density $\sigma$ is defined as the charge per unit area, i.e.,

$$\sigma = \frac{q}{A}$$

the electric field between the oppositely charged conducting plates can also be expressed as

$$E = \frac{\sigma}{\varepsilon_0}$$

(22.37)

Thus, the electric field between the oppositely charged conducting plates is also given by equation 22.37 in terms of the surface charge density $\sigma$ and the permittivity of free space $\varepsilon_0$, the medium between the plates. The configuration of the two oppositely charged conducting plates is called a parallel plate capacitor and we will return to this in chapter 25 when we discuss capacitors.

**Example 22.5**

The electric field between two oppositely charged conducting plates. Find the electric field between two oppositely charged circular conducting plates of 5.00 cm radius, if a charge of $8.70 \times 10^{-10}$ C is placed on the plates.

**Solution**

The area of the plate is

$$A = \pi r^2 = \pi(0.0500 \text{ m})^2 = 7.85 \times 10^{-3} \text{ m}^2$$

The electric field between the conducting plates is found from equation 22.36 as

$$E = \frac{q}{\varepsilon_0 A} = \frac{8.70 \times 10^{-10} \text{ C}}{(8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2)(7.85 \times 10^{-3} \text{ m}^2)}$$

$$E = 1.25 \times 10^4 \text{ N/C}$$

To go to this Interactive Example click on this sentence.

**The Language of Physics**

**Flux**

A quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface. In particular, the electric flux is defined as the
number of lines of the electric field intensity $E$ that pass normally through the surface area $A$ (p. ).

**Gauss’s law for electricity**
The electric flux $\Phi_E$ that passes through a surface surrounding electric charge $q$ is a measure of the amount of charge $q$ contained within the Gaussian surface. When $\Phi_E$ is a positive quantity, the Gaussian surface surrounds a source of positive charge, and electric flux diverges out of the surface. If the electric charge is negative then the electric field would go inward, through the Gaussian surface, to the electric charge. Thus, the flux passing through the Gaussian surface would be negative. Hence, whenever $\Phi_E$ is negative, the Gaussian surface surrounds a negative charge distribution and the electric flux converges into the Gaussian surface. If there is no enclosed charge, the electric flux $\Phi_E$ will be equal to zero. In this case, whatever electric flux enters one part of a Gaussian surface, the same amount must leave somewhere else (p. ).

**Summary of Important Equations**

**Linear charge density**

$$\lambda = \frac{q}{L}$$

**Surface charge density**

$$\sigma = \frac{q}{A}$$

**Volume charge density**

$$\rho = \frac{q}{V}$$

**Electric flux**

$$\Phi_E = E \cdot A = EA \cos \theta$$ (22.1)

**Electric flux**

$$\Phi_E = \oint d\Phi_E = \oint E \cdot dA$$ (22.6)

**Gauss’s law**

$$\Phi_E = \oint E \cdot dA = \frac{q}{\varepsilon_0}$$ (22.14)

**Gauss’s law**

$$\Phi_E = \oint E \cdot dA = \frac{1}{\varepsilon_0} \oint \rho \, dV$$ (22.16)

**Charge distributed over a volume**

$$q = \oint \rho \, dV$$ (22.15)

**Electric field outside of a uniform spherical charge distribution**

$$E = k \frac{q}{r^2}$$ (22.19)

**Electric field inside a uniform spherical charge distribution**

$$E = k \frac{r}{R^3} q$$ (22.26)

**Electric field of an infinite line of charge**

$$E = \frac{2k\lambda}{r}$$ (22.29)
Chapter 22: Electric Flux and Gauss’s Law

Electric field of an infinite sheet of charge \( E = \frac{\sigma}{2\varepsilon_0} \) (22.33)

Electric field inside a conductor \( E = 0 \) (22.35)

Electric field between two oppositely charged conducting plates \( E = \frac{q}{\varepsilon_0 A} \) (22.36)

Electric field between two oppositely charged conducting plates \( E = \frac{\sigma}{\varepsilon_0} \) (22.37)

Questions For Chapter 22

1. There is great similarity between the solution of the electric field in front of a charged conducting plate and the electric field between two oppositely charged conducting plates. Why are the electric fields the same? Could the single conducting plate be viewed as only one part of two oppositely charged conducting plates, but one is so far away it is effectively at infinity?

2. Under what conditions can we consider the electric field of a line of charge to be the same as the electric field of a cylinder of charge?

Problems For Chapter 22

22.2 Electric Flux

1. What is the magnitude of the electric flux \( \Phi_E \) emanating from a point charge of 2.00 \( \mu \)C?

2. Find the total flux \( \Phi_E \) passing through the sides of a cube 1.00 m on a side if a point charge of \( 5.00 \times 10^{-6} \) C is located at its center.

3. Find the electric flux \( \Phi_E \) through a hemisphere of radius \( R \) that is immersed in a uniform electric field directed in the \( z \)-direction as shown in the diagram.

Diagram for problem 3. 

Diagram for problem 4.
4. Find the total electric flux $\Phi_E$ through a cylindrical surface that is immersed in a uniform electric field $E$ that is parallel to the axis of the cylinder as shown in the diagram.

5. Find the electric flux $\Phi_E$ through a spherical surface of radius $R$ that contains an electric dipole as shown in the diagram.

6. The total electric flux $\Phi_E$ through a sphere is $2.82 \times 10^5$ N m$^2$/C. What is the value of the enclosed charge $q$?

7. A charge $q$ of 3.85 $\mu$C is at the center of a cube 2.54 cm on a side. Determine the electric flux $\Phi_E$ through one face of the cube.

8. The electric field intensity at the point $P$, a distance $r = 1.50$ m outside of a spherically charged body of radius $R = 0.45$ m, is $E = 1.40 \times 10^4$ N/C. Find the electric charge $q$ on the sphere and the surface charge density $\sigma$.

9. Find the electric field $E$ between the plates of a parallel plate capacitor of area $A = 2.00 \times 10^{-3}$ m$^2$ if a charge $q = 6.00$ $\mu$C is placed upon them.

10. A ring of charge of radius $R$ carries a total charge $q$. Find the electric field $E$ in the plane of the ring, at a distance $r$ from the center of the ring, for $r$ (a) inside the ring of charge and (b) outside the ring of charge. Indicate your assumptions.

11. A spherical shell of charge, of radius $R$, carries a total charge $q$. Find the electric field $E$ at a distance $r$ from the center of the shell, for $r$ (a) inside the spherical shell of charge and (b) outside the spherical shell of charge.
12. A point charge $q$ is placed at the center of two concentric conducting spheres. The radius of the inner sphere is $r_1$ while the radius of the outer sphere is $r_2$. Find the electric field $E$ for (a) $r < r_1$, (b) $r_1 < r < r_2$, and (c) $r > r_2$.

13. A positively charge rod is touched to the outside of a hollow spherical conductor of outer radius $r_b = 17.0$ cm and inner radius $r_a = 15.0$ cm, leaving a surface charge density $\sigma = 1.51 \times 10^{-5}$ C/m$^2$. Find the electric field $E$ for (a) $r < r_a$, (b) $r_a < r < r_b$, and (c) $r > r_b$.

14. Show that the total charge $q'$ contained within a spherical charge distribution, equation 22.23,

$$ q' = \frac{r^3}{R^3} q $$

can be derived from a more general approach by using

$$ q' = \int \rho \, dV $$

where $\rho$ is the constant volume charge density, and the element of volume of a sphere is $dV = 4\pi r^2 \, dr$.

15. A nonconducting sphere of charge, with a cavity in the center of the sphere, carries a uniform charge distribution $\rho$. Find the electric field $E$ for (a) $r < a$, (b) $a < r < b$, and (c) $r > b$. 
16. Two large sheets of charge (assume they are infinite) each carry a surface charge density $\sigma = 2.85 \times 10^{-5} \text{ C/m}^2$. Find the electric field intensity $E$ for (a) $0 < x < x_1$ (b) $x_1 < x < x_2$ (c) $x_2 < x$.

17. Repeat problem 16 only now the two sheets of charge are replaced by two conducting plates carrying the surface charge densities $\sigma_1 = 2.85 \times 10^{-5} \text{ C/m}^2$ and $\sigma_2 = -2.85 \times 10^{-5} \text{ C/m}^2$.

18. A thin hollow metal sphere of radius $R = 15.0 \text{ cm}$ carries a charge $q = 6.28 \times 10^{-6} \text{ C}$. Find the electric field intensity $E$ for (a) $r < R$ and (b) $r > R$. Let $r = 2.00 \text{ m}$.

19. Compare the electric field intensity $E$ a short distance $r$ outside a spherical conducting body with the electric field intensity $E$ the same distance $r$ outside a charged infinite conducting plate. Discuss the physical significance of the solution.

20. A conducting cylindrical shell of radius $R$ is placed around a very long line of charge of linear charge density $\lambda$. Find the electric field intensity $E$ for (a) $r < R$ and (b) $r > R$. 

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21. Two concentric spherical conducting shells have radii \( r_a \) and \( r_b \). A charge \(+q\) is placed on the inner spherical shell. Find (a) the charge \( q \) on the inside of the outer shell, and (b) the charge \( q \) on the outside of the outer shell. Also find the electric field intensity \( E \) for (c) \( r < r_a \), (d) \( r_a < r < r_b \), and (e) \( r > r_b \).

22. Repeat problem 21 only this time the outer shell is grounded.

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