Chapter 10  Gravitation - Planetary and Satellite Motion

10.1 Newton’s Law of Universal Gravitation
It is sometime said that Newton observed an apple fall from a tree and immediately discovered the law of gravity. This is, of course, an oversimplification. Newton did observe that when any object, such as an apple, was released near the surface of the earth, it was accelerated toward the earth. Since the cause of an acceleration is an unbalanced force, there must, therefore, be a force pulling objects toward the earth. If you throw the apple, or any other projectile, at an initial velocity $v_0$, as seen in figure 10.1, then instead of that apple moving off into space in a straight line as

![Figure 10.1](image1.png)

*Figure 10.1* Motion of an apple or a projectile.

Newton’s first law dictates, it is continually acted on by a force pulling it back to earth. If you were strong enough to throw the apple or any projectile with greater and greater initial velocities, then the projectile paths would be as shown in figure 10.2. The distance down range would become greater and greater until at some

![Figure 10.2](image2.png)

*Figure 10.2* The same force acting on a projectile acts on the moon.

initial velocity, the projectile would not hit the earth at all, but would go right around it in an orbit. But at any point along its path the projectile would still have a force acting on it pulling it down toward the surface of the earth just as it had in
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Figure 10.1. Figure 10.3 shows a page from the translated version of Newton’s *Principia* showing these ideas.

Newton extrapolated to the conclusion that the same force that causes the apple to fall to the earth also causes the moon to be pulled to the earth. Thus, the moon moves in its orbit about the earth because it is pulled toward the earth. It is falling toward the earth, just as the apple falls to the earth. If there is a force between the moon and the earth, why not a force between the sun and the earth? Or for that matter why not a force between the sun and all the planets? Newton proposed that the same gravitational force that acts on objects near the surface of the earth also acts on all the heavenly bodies. This was a revolutionary hypothesis at that time, for no one knew why the planets revolved around the sun. Following this line of reasoning to its natural conclusion, Newton proposed that there was a force of gravitation between each and every mass in the universe.

**Newton’s law of universal gravitation** was stated as follows: between every two masses in the universe there is a force of attraction between them that is directly proportional to the product of their masses, and inversely proportional to the square of the distance separating them. If the two masses are as shown in figure 10.4 with \( r \) the distance between the centers of the two masses, then the force of attraction is

\[
F = \frac{G m_1 m_2}{r^2}
\]

Figure 10.4 Newton’s law of universal gravitation.
\[ F = \frac{Gm_1m_2}{r^2} \]  

(10.1)

where \( G \) is a constant, called the universal gravitational constant, given by

\[ G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \]

We assume here that the radii of the masses are relatively small compared to the distance separating them so that the distance separating the masses is drawn to the center of the masses. Such masses are sometimes treated as point masses or particles. Spherical masses are usually treated like particles.

Note here that the numerical value of the constant \( G \) was determined by a celebrated experiment by Henry Cavendish (1731-1810) over 100 years after Newton’s statement of the law of gravitation. Cavendish used a torsion balance with known masses. The force between the masses was measured and \( G \) was then calculated.

**Gravitational Force between Two 1-Kg Masses**

Newton’s law of universal gravitation says that there is a force between any two masses in the universe. Let us set up a little experiment to test this law. Let us take two standard 1-kg masses and place them on the desk, so that they are 1 m apart, as shown in figure 10.5. According to Newton’s theory of gravitation, there is a force between these masses, and according to Newton’s second law, they should be accelerated toward each other. However, we observe that the two masses stay right where they are. They do not move together! Is Newton’s law of universal gravitation correct or isn’t it?

![Figure 10.5 Two 1-kg masses sitting on a table.](image-url)
Let us compute the gravitational force between these two 1-kg masses. We assume that the gravitational force acts at the center of each of the 1-kg masses. By equation 10.1, we have

\[ F = \frac{G m_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \text{ N m}^2}{\text{kg}^2 \text{(1 m)}^2} \]

and therefore the force acting between these two 1-kg masses is

\[ F = 6.67 \times 10^{-11} \text{ N} \]

This is, of course, a very small force. In fact, if this is written in ordinary decimal notation we have

\[ F = 0.0000000000667 \text{ N} \]

A very, very small force indeed. (Sometimes it is still worth while for the beginning student to write numbers in this ordinary notation to get a better “feel” for the meaning of the numbers that are expressed in scientific notation.)

If we redraw figure 10.5 showing all the forces acting on the masses, we get figure 10.6. The gravitational force on mass \( m_2 \) is trying to pull it toward the left.

\[ F_{N_1} \]
\[ F_{N_2} \]
\[ F_g \]
\[ w_1 \]
\[ w_2 \]
\[ r = 1 \text{ m} \]

**Figure 10.6** Gravitational force on two 1-kg masses.

But if the body tends to slide toward the left, there is a force of static friction that acts to oppose that tendency and acts toward the right. The frictional force that must be overcome is

\[ f_s = \mu_s F_{N_2} = \mu_s w_2 = \mu_s m_2 g \]

Assuming a reasonable value of \( \mu_s = 0.50 \) we obtain for this frictional force,


\[ f_s = \mu_s m_2 g = (0.50)(1.00 \text{ kg})(9.80 \text{ m/s}^2) = 4.90 \text{ N} \]

Hence, to initiate the movement of the 1-kg mass across the table, a force greater than 4.90 N is needed. As you can see, the gravitational force \((6.67 \times 10^{-11} \text{ N})\) is nowhere near this value, and is thus not great enough to overcome the force of static friction. Hence you do not, in general, observe different masses attracting each other. That is, two chairs do not slide across the room and collide due to the gravitational force between them.

If these small 1-kg masses were taken somewhere out in space, where there is no frictional force opposing the gravitational force, the two masses would be pulled together. It will take a relatively long time for the masses to come together because the force, and hence the acceleration is small, but they will come together within a few days.

**Gravitational Force between a 1-Kg Mass and the Earth**

The reason why the computed gravitational force between the two 1-kg masses was so small is because \(G\), the universal gravitational constant, is very small compared to the masses involved. If, instead of considering two 1-kg masses, we consider one mass to be the 1-kg mass and the second mass to be the earth, then the force between them is very noticeable. If you let go of the 1-kg mass, the gravitational force acting on it immediately pulls it toward the surface of the earth. The cause of the greater force in this case is the larger mass of the earth.

**Example 10.1**

*The gravitational force between a 1-kg object and the earth.* Determine the gravitational force on a 1-kg mass near the surface of the earth.

**Figure 10.7** Gravitational force on a 1-kg mass near the surface of the earth.

**Solution**
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Figure 10.7 shows a mass \( m_1 \) of 1 kg a distance \( h \) above the surface of the earth. The radius of the earth \( r_e \) is \( r_e = 6.371 \times 10^6 \) m, and its mass \( m_e \) is \( m_e = 5.977 \times 10^{24} \) kg. The separation distance between \( m_1 \) and \( m_e \) is

\[
r = r_e + h \approx r_e
\]

since \( r_e \gg h \). The gravitational force acting on that 1-kg mass is

\[
F_g = \frac{Gm_em_1}{r_e^2}
\]

\[
= \left( \frac{6.67 \times 10^{-11} \text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \frac{(5.977 \times 10^{24} \text{ kg})(1.00 \text{ kg})}{(6.371 \times 10^6 \text{ m})^2}
\]

\[
= 9.82 \text{ N}
\]

To go to this Interactive Example click on this sentence.

This number should be rather familiar. Recall that the weight of a 1.00-kg mass was determined from Newton’s second law as

\[
w = mg = (1.00 \text{ kg})(9.80 \text{ m/s}^2) = 9.80 \text{ N}
\]

(The standard value of \( g = 9.80 \text{ m/s}^2 \) has been used. We will see shortly that \( g \) can actually vary between 9.78 m/s\(^2\) at the equator to 9.83 m/s\(^2\) at the pole. Also the radius of the earth \( r_e \) used in equation 10.3 is the mean value of \( r_e \). The actual value of \( r_e \) varies slightly with latitude.)

The point to notice here is that the weight of a body is in fact the gravitational force acting on that body by the earth and pulling it down toward the center of the earth. Thus, the weight of a body is actually determined by Newton’s law of universal gravitation. This points up even further the difference between the mass and the weight of a body.

**Newton’s law of universal gravitation in vector form.**

In equation 10.1, we wrote Newton’s law of universal gravitation in terms of its magnitude only. Newton’s law of gravitation can be written in a more general form as follows. Let \( m_1 \) be the primary mass that we are interested in. We will introduce a unit vector \( \mathbf{r}_o \) that points everywhere radially away from the primary mass \( m_1 \) as shown in figure 10.8.

If a second mass \( m_2 \) is brought into the vicinity of mass \( m_1 \), it will experience the force

\[
F_{21} = -G \frac{m_1 m_2}{r^2} \mathbf{r}_o
\]

where \( F_{21} \) is the force on mass \( m_2 \) caused by mass \( m_1 \). The force on mass \( m_2 \) is in the
opposite direction of the unit vector \( \mathbf{r}_o \), and the force is one of attraction as seen in figure 10.8. By Newton’s third law, the force on mass \( m_1 \) is equal and opposite to the force on mass \( m_2 \) as expected. That is,

\[
F_{12} = - F_{21}
\]

10.2 The Acceleration Due to Gravity and Newton’s Law of Universal Gravitation

Newton’s second law states that when an external unbalanced force acts on an object, it will give that object an acceleration \( a \), that is,

\[
F = ma
\]

But if the force acting on a body near the surface of the earth is the gravitational force, then that body experiences the acceleration \( g \) of a freely falling body, as shown in section 2.7. That is, the force acting on the object is called its weight, and it experiences the acceleration \( g \). Thus, Newton’s second law becomes

\[
mg = F_g
\]

Using equations 10.5 and 10.3 we get

\[
mg = \frac{Gm_1m_2}{r_e^2}
\]

Solving for \( g \), we obtain

\[
g = \frac{Gm_1}{r_e^2}
\]

That is, the acceleration due to gravity, \( g \), which in chapter 2 was accepted as an experimental fact, can be deduced from theoretical considerations of Newton’s second law and his law of universal gravitation.
Example 10.2

Determine the acceleration of gravity \( g \) at the surface of the earth.

Solution

The value of \( g \), determined from equation 10.7, is

\[
g = \frac{G m_e}{r_e^2} = \left( \frac{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}}{\text{kg}^2} \right) \left( \frac{5.977 \times 10^{24} \text{ kg}}{(6.371 \times 10^6 \text{ m})^2} \right) = 9.82 \text{ m/s}^2
\]

To go to this Interactive Example click on this sentence.

Newton introduced his law of universal gravitation, and a by-product of it is a theoretical explanation of the acceleration due to gravity \( g \). This is an example of the beauty and simplicity of physics. There is no way that we could have predicted the relation of equation 10.7 from purely experimental grounds. Yet Newton’s second law and his law of universal gravitation, in combination, have made that prediction.

10.3 Variation of the Acceleration Due to Gravity

We can see from equation 10.7 one of the reasons why the acceleration due to gravity \( g \) is very nearly a constant. \( G \) is a constant and \( m_e \) is a constant, but \( r_e \) is not exactly a constant. The earth is not, in fact, a perfect sphere. It is, rather, an oblate spheroid. That is, the radius of the earth at the equator \( r_{ee} \) is slightly greater than the radius of the earth at the poles \( r_{ep} \), as seen in figure 10.9. The diagram is,

![Figure 10.9](image)

*Figure 10.9* The earth is an oblate spheroid.

of course, exaggerated to show this difference. The actual values of \( r_{ee} \) and \( r_{ep} \) are

\[
r_{ee} = 6.378 \times 10^6 \text{ m}
\]
with the mean radius

\[ r_e = 6.371 \times 10^6 \text{ m} \]

The variation in the radius of the earth is thus quite small. However, the variation, although small, does contribute to the observed variation in the acceleration due to gravity on the earth from a low of 9.78 m/s\(^2\) at the equator to a high of 9.83 m/s\(^2\) at the North Pole, as seen in table 10.1. This analysis also assumes that the earth is not rotating.

The effect of the rotation of the earth on the acceleration due to gravity.

A more sophisticated analysis of the acceleration due to gravity takes into account the variation in \(g\) caused by the centripetal acceleration, which varies with latitude on the surface of the earth. The acceleration due to gravity \(g_e\) for a nonrotating earth is given by equation 10.7 and it points toward the center of the earth as shown in figure 10.10(a). Because the earth is rotating there must also be a centripetal acceleration \(a_c\) that points toward the axis of rotation of the earth as is shown in figure 10.10(a). Recall that a centripetal force is not a real force that is acting on a rotating body, but is the force that must be applied to keep the body in that rotational motion. The total force acting on a body at the surface of the rotating earth will be equal to the difference between the gravitational force of the earth and the centripetal force. (Another way to look at this is to note that part of the gravitational force must supply the centripetal force to keep the body rotating on the surface of the earth.) Hence, the resultant force on a body at the surface of the earth is the vector difference of these two forces, that is,

\[ \mathbf{F} = \mathbf{F}_e - \mathbf{F}_c \]
and thus the resultant gravitational acceleration is the vector difference between these two similar acceleration terms, that is,

\[ g = g_e - a_c \quad (10.8) \]

To determine the magnitude of \( g \) let us take the dot product of equation 10.8 with itself, that is,

\[ g \cdot g = (g_e - a_c) \cdot (g_e - a_c) = g_e \cdot g_e + a_c \cdot a_c - 2 a_c \cdot g_e \]

and using the definition of the dot product this becomes

\[ g^2 = g_e^2 + a_c^2 - 2a_c g_e \cos \phi \]

Upon taking the square root of both sides of the equation we get

\[ g = \sqrt{g_e^2 + a_c^2 - 2a_c g_e \cos \phi} \quad (10.9) \]

\textit{Equation 10.9 gives the acceleration due to gravity at a particular latitude \( \phi \) taking the rotation of the earth into account.} The centripetal acceleration is a function of the latitude angle \( \phi \) and as seen in figure 10.10(b) is given by

\[ a_c = \omega^2 r = \omega^2 R \cos \phi \quad (10.10) \]

where \( R \) is the radius of the earth. The acceleration due to gravity on a nonrotating earth \( g_e \) is given by equation 10.7.

**Example 10.3**

The acceleration due to gravity taking the earth’s rotation into account. Determine the acceleration due to gravity at 40.0\(^0\) N latitude taking the earth’s rotation into account.

**Solution**

The angular velocity \( \omega \) of the earth is \( 7.27 \times 10^{-5} \) rad/s, and its mean radius is \( R = 6.37 \times 10^6 \) m. The centripetal acceleration \( a_c \) is found from equation 10.10 as

\[ a_c = \omega^2 R \cos \phi \]

\[ a_c = (7.27 \times 10^{-5} \text{ rad/s})^2 (6.37 \times 10^6 \text{ m}) \cos 40.0^0 \]

\[ a_c = 0.0258 \text{ m/s}^2 \]

The acceleration due to gravity on a nonrotating earth \( g_e \) is given by equation 10.7 as
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\[ g_e = \frac{Gm_e}{r_e^2} \]
\[ = \left( \frac{6.67 \times 10^{-11} \text{N m}^2}{\text{kg}^2} \right)(5.977 \times 10^{24} \text{ kg}) \]
\[ (6.37 \times 10^6 \text{ m})^2 \]
\[ = 9.82 \text{ m/s}^2 \]

The acceleration due to gravity on a rotating earth is found from equation 10.9 as

\[
g = \sqrt{g_e^2 + a_c^2 - 2a_c g_e \cos \phi}
\]

\[
g = \sqrt{(9.82 \text{ m/s}^2)^2 + (0.0258 \text{ m/s}^2)^2 - 2(0.0258 \text{ m/s}^2)(9.82 \text{ m/s}^2) \cos 40.00^\circ}
\]

\[ g = 9.80 \text{ m/s}^2 \]

To go to this Interactive Example click on this sentence.

The standard value of \( g \), adopted for most calculations in physics, is

\[ g = 9.80 \text{ m/s}^2 \]

the value at 45° north latitude at the surface of the earth.

At greater heights, \( g \) also varies slightly from that given in equation 10.7 because of the approximation

\[ r_e \approx r_e + h \]

that was made for that equation. Although this approximation is, in general, quite good for most locations, if you are on the top of a mountain, such as Pikes Peak, this higher altitude (large value of \( h \)) will give you a slightly smaller value of \( g \), as we can see in table 10.1.

<table>
<thead>
<tr>
<th>Table 10.1</th>
<th>Different Values of ( g ) on the Earth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>Value of ( g ) in m/s²</td>
</tr>
<tr>
<td>Equator at sea level</td>
<td>9.78</td>
</tr>
<tr>
<td>New York City</td>
<td>9.80</td>
</tr>
<tr>
<td>45° N latitude (standard)</td>
<td>9.80</td>
</tr>
<tr>
<td>North Pole</td>
<td>9.83</td>
</tr>
<tr>
<td>Pikes Peak - elevation 4293 m</td>
<td>9.79</td>
</tr>
<tr>
<td>Denver, Colorado - elevation 1638 m</td>
<td>9.80</td>
</tr>
</tbody>
</table>

Again it is quite remarkable that these slight variations in the observed experimental values of \( g \) on the surface of this earth can be explained and predicted by Newton’s law of universal gravitation, with slight corrections for the radius of the earth, the centripetal acceleration (which is a function of latitude), and the height of the location above mean sea level. There are also slight local variations in
$g$ due to the nonhomogeneous nature of the mass distribution of the earth. These variations in $g$ due to different mass distributions are used in geophysical explorations. One of the many scientific experiments performed on the moon was a mapping of the acceleration due to gravity on the moon to disclose the possible locations of different mineral deposits.

### 10.4 Acceleration Due to Gravity on the Moon and on Other Planets

Equation 10.7 was derived on the basis of the gravitational force of the earth acting on a mass near the surface of the earth. The result is perfectly general however. If, for example, an object were placed close to the surface of the moon, as shown in figure 10.11, the force on that mass would be its lunar weight, which is just the gravitational force of the moon acting on it. Therefore the weight of an object on the moon is

$$w_m = F_g$$

This becomes

$$mg_m = \frac{Gm_m m}{r_m^2}$$

where $g_m$ is the acceleration due to gravity on the moon and $m_m$ and $r_m$ are the mass and radius of the moon, respectively. Hence, the acceleration due to gravity on the moon is

$$g_m = \frac{Gm_m}{r_m^2} \quad (10.11)$$

Equation 10.11 is identical to equation 10.7 except for the subscripts. Therefore, we can use equation 10.7 to determine the acceleration due to gravity on any of the planets, simply by using the mass of that planet and the radius of that planet in equation 10.7.
Example 10.4

Gravity on the moon. Determine the acceleration due to gravity on the moon.

Solution

The acceleration due to gravity on the moon, found by solving equation 10.11, is

\[ g_m = \frac{G m_m}{r_m^2} = \left( \frac{6.67 \times 10^{-11} \text{ N m}^2}{\text{kg}^2} \right) \left( \frac{7.34 \times 10^{22} \text{ kg}}{1.738 \times 10^6 \text{ m}^2} \right) = 1.62 \text{ m/s}^2 = 0.165 \ g_e \approx \frac{1}{6} \ g_e \]

The acceleration due to gravity on the moon is approximately 1/6 the acceleration due to gravity on the earth.

To go to this Interactive Example click on this sentence.

Because the weight of an object is \( w = mg \)

the weight of an object on the moon is

\[ w_m = m g_m = m \left( \frac{1}{6} g_e \right) = \frac{1}{6} \ (m g_e) = \frac{1}{6} \ w_e \]

which is 1/6 of the weight that it had on the earth. That is, if you weigh 180 lb on earth, you will only weigh 30 lb on the moon.

Table 10.2 is a list of the masses, radii, and values of \( g \) on the various planets. Note that the most massive planet is Jupiter, and it has an acceleration due to gravity of

\( g_J = 2.37 \ g_e \)

Therefore, the weight of an object on Jupiter will be

\[ w_J = 2.37 \ w_e \]

If you weighed 180 lb on earth, you would weigh 427 lb on Jupiter.
10.5 Generalization of the Formulation for the Gravitational Potential Energy

In section 7.5 we found the gravitational potential energy of an object that was at some height \( h \) above the surface of the earth. We showed that because work must be done on a body to put the body into the position where it has potential energy, the work done was used as the measure of this potential energy. That is, the potential energy of a body is equal to the work done to put the body into the particular position. We showed in equation 7.19 that the potential energy (PE) is

\[
PE = W = \int dW = \int F \cdot ds
\]  

(7.19)

In equation 7.19 we applied a constant force \( F = mg \) to lift the object from the ground to a height \( h \) and found that

\[
PE = \int_0^h mgy = [mgy]_0^h
\]

We took the lower limit of integration to be 0, the location of the mass when it is on the ground. This meant that the zero of potential energy was at the ground. The upper limit was \( h \), the height of the mass when it was lifted to its highest position, where the object then had the potential energy given by equation 7.24 as

\[
PE = mgh
\]  

(7.24)

Our derivation of the potential energy assumed that there was an applied force acting on the body, and that applied force was a constant. For the general case of gravitational potential energy we consider two spheres of mass \( m_1 \) and \( m_2 \) in
space such as shown in figure 10.12. We say that the combination of two spheres contain potential energy in their present configuration because both are capable of doing work, when the gravitational attraction causes them to move together. To determine the potential energy we use a slightly different approach than we did in chapter 7. In chapter 7 we computed the work done by an applied external force that lifted the mass to the height $h$. Since we are not capable of applying an external force to move the two large masses, we consider the work done by the gravitational force itself, rather than an applied external force exerted by us, when the mass $m_2$ moves from position $A$ to position $B$. In this case we will see that the work done by the gravitational force will be negative, because we would have to do positive work to prevent the two masses from moving together. Also since the gravitational force varies with $r$ the force is not a constant. Taking both considerations into account we find the change in the potential energy $d(PE)$ for

$$d(PE) = dW$$

Calling the potential energy at the point $A$, $PE_A$ and at $B$, $PE_B$ the total change in potential energy between the points $A$ and $B$ is found by adding or integrating all the $dW$s that are between $A$ and $B$, that is

$$\int_{PE_A}^{PE_B} d(PE) = \int dW = \int \mathbf{F} \cdot d\mathbf{s}$$

(10.12)
But the force $F$ is the gravitational force on mass 2 caused by mass 1 and is designated as $F_{21}$, that is,

$$\text{PE}_B - \text{PE}_A = \int F_{21} \cdot ds$$  \hspace{1cm} (10.13)

For this particular case the force $F_{21}$ is the gravitational force on mass 2 caused by mass 1 and is given by

$$F_{21} = - \frac{G m_1 m_2}{r^2} \mathbf{r}_0$$  \hspace{1cm} (10.14)

where $\mathbf{r}_0$ is the unit vector pointing from $m_1$ to $m_2$. We have a slight problem here because $F_{21}$ is a function of $r$ and yet the integration is over the path $s$. But $ds$ and $dr$ are related as can be seen in figure 10.12(b). The coordinate vector $\mathbf{r}$ is measured from the mass $m_1$ outward. Its value at $B$ is $\mathbf{r}_B$ and at $A$, $\mathbf{r}_A$ where $\mathbf{r}_A > \mathbf{r}_B$. The element of $\mathbf{r}$ is the final value minus the initial value or

$$d\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$$

The element of path $ds$, on the other hand, extends from $A$ to $B$ and since it is again found as the final value minus the initial value it is given by

$$ds = \mathbf{r}_B - \mathbf{r}_A$$

Comparing these equations we see that the displacement $ds$, as the mass is moved from position $A$ to position $B$, is in the negative $\mathbf{r}$ direction. That is

$$ds = -d\mathbf{r} = -\mathbf{r}_0 \, dr$$  \hspace{1cm} (10.15)

where $\mathbf{r}_0$ is the unit vector pointing from the mass $m_1$ to the mass $m_2$. Replacing equation 10.14 and 10.15 into equation 10.13 yields

$$\text{PE}_B - \text{PE}_A = - \int_A^B F_{21} \cdot d\mathbf{r} = - \int_A^B \left( - \frac{G m_1 m_2}{r^2} \mathbf{r}_0 \right) \cdot \mathbf{r}_0 \, dr$$  \hspace{1cm} (10.16)

Recall that the dot product of the unit vectors $\mathbf{r}_0 \cdot \mathbf{r}_0 = 1$. Equation 10.16 becomes

$$\text{PE}_B - \text{PE}_A = \int_{r_A}^{r_B} \frac{G m_1 m_2}{r^2} \, dr$$  \hspace{1cm} (10.17)

Notice that the value of the limits of $r$ at $A$ and $B$ are $r_A$ and $r_B$ respectively and are now the limits of integration in equation 10.17. Evaluating the integral we obtain

$$\text{PE}_B - \text{PE}_A = \int_{r_A}^{r_B} \frac{G m_1 m_2}{r^2} \, dr = G m_1 m_2 \int_{r_A}^{r_B} r^{-2} \, dr = G m_1 m_2 \left[ -\frac{1}{r^{-2}} \right]_{r_A}^{r_B}$$
Equation 10.18 gives the difference in potential energy between the point A and the point B in the gravitational field of a mass $m_1$. In chapter 7 we set the zero of potential energy to be at the surface of the earth. We now let the zero of potential energy to be at infinity. That is, at $r_A = \infty$, $PE_A = 0$, and we drop the subscript $B$ on $r_B$ so that it refers to any point $r$. Equation 10.18 then becomes

$$PE = -\frac{Gm_1 m_2}{r}$$

(10.19)

Equation 10.19 gives the potential energy of a mass $m_1$ a distance $r$ from the mass $m_2$. Notice that the potential energy is a negative quantity.

**Example 10.5**

Potential energy of the earth-moon system. Assuming that the orbit of the moon about the earth is a circle of radius $r = 3.84 \times 10^8$ m, find the potential energy of the earth-moon system.

**Solution**

The potential energy of the earth-moon system is found by solving equation 10.19, as

$$PE = -\frac{Gm_1 m_2}{r}$$

$$= -\left(\frac{6.67 \times 10^{-11} \text{ N m}^2}{\text{kg}^2}\right)\frac{(5.97 \times 10^{24} \text{ kg})(7.34 \times 10^{22} \text{ kg})}{(3.84 \times 10^8 \text{ m})}$$

$$= 7.61 \times 10^{28} \text{ N m} = 7.61 \times 10^{28} \text{ J}$$

Because of the large masses of both the earth and the moon, the potential energy of the earth-moon system is rather large, as you can see.
Example 10.6

**Difference in gravitational potential energy.** Find the difference in potential energy of a satellite of mass $m_s = 2.50 \times 10^3$ kg when it is at a distance of the moon and when it is at the distance of the planet mars.

**Solution**

The distance from earth to the moon is $r_B = 3.84 \times 10^8$ m, and assume the distance from the earth to the planet mars is approximately $r_A = 0.78 \times 10^{11}$ m. The difference in potential energy of the satellite is found by solving equation 10.18, as

$$PE_B - PE_A = -\frac{Gm_E m_s}{r_B} - \frac{1}{r_A}$$

$$= -\left(\frac{6.67 \times 10^{-11} \text{N m}^2}{\text{kg}^2}\right)(5.97 \times 10^{24} \text{ kg})(2.50 \times 10^3 \text{ kg})\left(\frac{1}{3.84 \times 10^8 \text{ m}} - \frac{1}{0.78 \times 10^{11} \text{ m}}\right)$$

$$= -2.58 \times 10^8 \text{ N m} = -2.58 \times 10^9 \text{ J}$$

**To go to this Interactive Example click on this sentence.**

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**The Velocity of Escape**

Suppose that we wished to fire a rocket from the surface of the earth that would completely escape the gravitational force of the earth and would be free to leave the earth. What velocity would we have to impart to such a rocket? Let us apply the law of conservation of energy to the rocket, that is,

$$E = KE + PE = \frac{1}{2} m_r v^2 - \frac{Gm_r M}{r}$$

(10.20)

where $m_r$ is the mass of the rocket and $M$ the mass of the earth. For the rocket to escape the gravitational force of the earth it must have zero speed and zero energy when it reaches infinity. The total energy of the rocket at the surface of the earth must equal the total energy of the rocket when it reaches infinity. Designating the velocity of escape as $v_e$, the radius of the earth as $R_e$, and equating the total energy of the rocket at the surface of the earth to the total energy of the rocket at infinity we get

---

1\(^*\) Actually for the rocket to escape from the earth’s gravitational field, its energy must be greater than zero when it reaches infinity. We assume that the energy of the rocket at infinity is infinitesimally greater than zero. So we again use the standard technique of letting the energy $E = 0$. This is the absolute minimum energy the rocket must have. Any value greater than this will allow the rocket to escape, any value less than this will cause the rocket to be pulled back to the earth by the gravitational force.
Chapter 10  Gravitation - Planetary and Satellite Motion

\[
\frac{1}{2} m_r v^2 - \frac{G m_r M}{R_e} = \frac{1}{2} m_r (0)^2 - \frac{G m_r M}{\infty}
\]

\[
\frac{1}{2} m_r v^2 = \frac{G m_r M}{R_e}
\]

and

\[
v_e = \sqrt{\frac{2GM}{R_e}}
\]  \hspace{1cm} (10.21)

*Equation 10.21 gives the magnitude of the velocity of escape for a rocket to leave the surface of the earth and not be pulled back by the gravitational force of the earth.*

The direction of the velocity vector is, of course, perpendicular to the surface of the earth.

**Example 10.7**

*Escape velocity.* Find the velocity necessary for a rocket to escape from the gravitational field of the earth.

**Solution**

The escape velocity of the rocket from the surface of the earth is found by solving equation 10.21, as

\[
v_e = \sqrt{\frac{2GM}{R_e}}
\]

\[
= \sqrt{\frac{2 \left( 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \right) \left( 5.97 \times 10^{24} \text{ kg} \right)}{(6.37 \times 10^6 \text{ m})}}
\]

\[
= 1.12 \times 10^4 \text{ m/s} = 11.2 \text{ km/s}
\]

Hence a rocket would have to be given a speed of 11.2 km/s, perpendicular to the surface of the earth, for it to escape the gravitational force of the earth.

To go to this Interactive Example click on this sentence.

A similar analysis for the escape velocity from any planet can be found by using equation 10.21 with \( M \) as the mass of that planet and \( R \) as the radius of that planet. Similarly to determine the escape velocity from our solar system, \( M \) would be the mass of the sun and \( R \) would be its radius.
10.6 Planetary Motion

The study of the motion of the planets is perhaps one of the oldest of all the sciences. In those very dark nights of the early days of the history of mankind, people could not help but notice the sky. They observed a regularity in the movements of the sun, moon, and stars, and asked what is the cause of the regularity in the motion of these heavenly bodies? What makes the sun rise, move across the sky, and then set? What makes the stars and moon move in the night sky? From these early observations of the sky it was assumed that the earth was the center of the universe and the sun, moon, and all the planets revolved around the earth. This was not an unrealistic assumption, for every day you see the sun rise in the east and set in the west. It certainly appears as though the sun is revolving about the earth. This model of the universe was called the Ptolemaic system and dates back to Claudius Ptolemy (100-170). It was also referred to as a geocentric system of the universe. One of the problems of this geocentric system was found when observing the night sky. Certain apparent stars seemed to move across the sky and then stop and move backward for a short time and then start moving forward again, their path forming a small loop in the night sky. These “wandering stars” were called planets, from the Greek word for wanderer. The geocentric system could not give a good explanation for this strange motion of the planets, though many people tried. It was not till 1543 that the Polish astronomer, Nicholas Copernicus (1473-1543) put forth a new model of the universe that stated that the earth and the other planets revolved about the sun. This heliocentric system was defended by Galileo. Extremely precise experimental determinations of the orbits of the planets were made by the Danish astronomer Tycho Brahe (1546-1609). Johannes Kepler (1571-1630) analyzed this work and expressed the result in what are now called Kepler’s laws of planetary motion. Kepler’s laws are

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The speed of the planet varies in such a way that the line joining the planet and the sun sweeps out equal areas in equal times.
3. The square of the time for the planet to make a complete revolution about the sun is proportional to the cube of the semimajor axes of the elliptical orbit.

Kepler’s First Law of planetary motion

Kepler’s first law states that the orbit of each planet is an ellipse with the sun at one focus. This can be seen in figure 10.13. Notice that the sun is at one focus of the ellipse and the planet is moving along the elliptical path. The distance from the sun to the planet at any time is given by the distance $r$. Again note that this distance keeps changing as the planet moves in its orbit. When the planet is at its closest approach to the sun, $r = r_p$ in the diagram, the planet is said to be at its perihelion position. When the planet is at its furthest position from the sun, $r = r_a$ in the diagram, the distance is called the aphelion distance. The distance $a$ in figure 10.13 is called the semimajor axis of the ellipse, and can be expressed in terms of the perihelion and aphelion distances by observing from the figure that
For the special case where the ellipse degenerates into a circle, $r_a = r_p = r$ the radius of the circular orbit, and then

$$a = \frac{r_a + r_p}{2} = \frac{r + r}{2} = \frac{2r}{2} = r$$

Hence the circular orbit is only a special case of the elliptical orbit. A measure of the ellipticity of the orbit is given by the eccentricity, which is defined as

$$e = \frac{r_a - r_p}{2a}$$

In general, since $r_a - r_p < 2a$, the eccentricity of an ellipse is less than 1. For a circular orbit $r_a = r_p = r$, and the eccentricity then becomes

$$e = \frac{r_a - r_p}{2a} = \frac{r - r}{2r} = 0$$

Thus for a circular orbit, the eccentricity is equal to zero. In general, the eccentricity varies from 0 to 1 for an elliptical orbit.

**Circular Orbits**

The orbits of all the planets around the sun are ellipses. But, in general, the amount of ellipticity is relatively small, and as a first approximation it is quite often assumed that their orbits are circular, because the analysis of a circular orbit is quite simple. Let us now consider the motion of a planet around the sun in a circular orbit. For the planet to be in motion in a circular orbit, there must be a
centripetal force acting on the planet to force it into the circular motion. This centripetal force acting on the planet, is supplied by the force of gravity of the sun. Let us assume that the planet is in an orbit a distance \( r \) from the center of the sun, as shown in figure 10.14. Because the centripetal force is supplied by the gravitational force, we have

\[
F_c = F_g \tag{10.24}
\]

or

\[
m_p v^2 = \frac{G m_s m_p}{r^2} \tag{10.25}
\]

\[\text{Figure 10.14 Planet in a circular orbit.}\]

The first thing that we note is that \( m_p \), the mass of the planet, divided out of equation 10.25. This means that the speed of the planet in its orbit will be independent of its mass. Solving for the speed of the planet in the circular orbit, we obtain

\[
v = \sqrt{\frac{G m_s}{r}} \tag{10.26}
\]

\[\text{Equation 10.26 represents the speed of a planet in a circular orbit, at a distance } r \text{ from the center of the sun. Equation 10.26 also says that the speed depends only on the radius of the orbit } r \text{, and the mass } m_s \text{ of the sun. For large values of } r \text{, the speed of the planet will be relatively small; whereas for small values of } r \text{, the speed } v \text{ of the planet will be much larger.}\]

\[\text{Example 10.8}\]

\[\text{Speed of the earth in a circular orbit. Determine the speed of the earth, in the circular orbit about the sun, shown in figure 10.15.}\]

\[\text{Figure 10.15 The speed of the earth in its orbit about the sun.}\]
The mass of the sun is \( m_s = 1.99 \times 10^{30} \) kg. The mean orbital radius of the earth around the sun, found in table 10.2, is \( r_{es} = 1.50 \times 10^{11} \) m. The speed of the earth around the sun \( v_{es} \), found from equation 10.26, is

\[
v_{es} = \sqrt{\frac{Gm_s}{r_{es}}} = \sqrt{\frac{6.67 \times 10^{-11} \text{N} \text{m}^2}{\text{kg}^2} \times \frac{1.99 \times 10^{30} \text{kg}}{1.50 \times 10^{11} \text{m}}} = 2.97 \times 10^4 \text{ m/s} = 29.7 \text{ km/s} = 66600 \text{ mi/hr}
\]

That is, as you sit and read this book, you are speeding through space at 66,600 mph. This is a little over 18 miles each second. The mean orbital speed of any of the planets can be determined in the same way.

To go to this Interactive Example click on this sentence.

To determine the approximate speed of any of the planets about the sun, we can use equation 10.26, with the appropriate radius of the orbit of the planet. For precise astronomical work, however, the elliptical orbit must be used.

Kepler’s Second Law of planetary motion

Kepler’s second law of planetary motion states that the speed of the planet varies in such a way that the line joining the planet and the sun sweeps out equal areas in equal times. This is illustrated in figure 10.16. At the time \( t_1 \) the planet is at the position P. As the planet moves to the point P’ at the time \( t_1’ \) the radius vector from the sun to the planet sweeps out the shaded area \( A_1 \) in the time interval \( t_1’ - t_1 \). Later when the planet is at the position Q at the time \( t_2 \), it moves to the point Q’ at the time \( t_2’ \). As the planet moves to the point Q’ at the time \( t_2’ \) the radius vector from the sun to the planet sweeps out the shaded area \( A_2 \) in the time interval \( t_2’ - t_2 \),
where $t_2' - t_2 = t_1' - t_1$. Kepler’s second law says that the planet sweeps out the area $A_1$ in the same time as it sweeps out the area $A_2$, and that $A_1 = A_2$. The most important consequence of these equal areas in equal times is that the speed of the planet is not a constant in an elliptical orbit as it was in a circular orbit. You can see in figure 10.16 that for the planet to move along the long path from $P$ to $P'$ in the same time interval that it moves along the short path from $Q$ to $Q'$, the planet must move much faster along the path $P-P'$, than it does along the path $Q-Q'$. Thus the planet is moving much faster at perihelion than it does at aphelion.

Kepler’s second law is really a consequence of the law of conservation of angular momentum as we will now show. Consider a planet located at the position $P$ by the position vector $\mathbf{r}$ in figure 10.17 and moving through the displacement $d\mathbf{s}$ as it moves toward the position $P'$. The angular momentum of the planet with respect to the sun is given by

$$L = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m \mathbf{v} \tag{10.27}$$

The velocity $\mathbf{v} = d\mathbf{s}/dt$, and equation 10.27 can be rewritten as

$$\frac{L}{m} = \mathbf{r} \times \frac{d\mathbf{s}}{dt} = \mathbf{r} \times d\mathbf{A} \tag{10.28}$$

But recall from section 3.12 The Vector Product, that the vector product of two vectors is equal to the area of the parallelogram that is formed by the vectors (see equation 3.57). Notice from figure 10.17 that the vector product

$$\mathbf{r} \times d\mathbf{s} = 2 d\mathbf{A} \tag{10.29}$$

Replacing equation 10.29 into equation 10.28 gives

$$\frac{L}{m} = \mathbf{r} \times \frac{d\mathbf{s}}{dt} = 2 \frac{d\mathbf{A}}{dt} \tag{10.30}$$

or
Equation 10.31 says that the rate at which the area swept out with time is equal to the angular momentum of the planet divided by 2m. But Newton’s second law for rotational motion was shown to be \( \tau = d\mathbf{L}/dt \). Since there is no torque acting on the planet, \( \tau = 0 \) and hence \( d\mathbf{L}/dt = 0 \) which means that the angular momentum \( \mathbf{L} \) is equal to a constant. Equation 10.31 thus becomes

\[
\frac{d\mathbf{A}}{dt} = \frac{\mathbf{L}}{2m} = \text{constant} \tag{10.32}
\]

Equation 10.32 says that the rate at which the area changes with time is equal to a constant, which means the planet sweeps out equal areas \( d\mathbf{A} \) in equal times \( dt \). But this is, of course, just Kepler’s second law. Hence, Kepler’s second law is a consequence of the law of conservation of angular momentum applied to the planet in its orbit about the sun.

In figure 10.16 we showed that the consequence of equal areas in equal times from Kepler’s second law is that the speed of the planet in an elliptical orbit is not constant. We can see this even more clearly by using the law of conservation of angular momentum in its scalar form, that is

\[
L = |\mathbf{r} \times m\mathbf{v}| = rmv \sin \theta \tag{10.33}
\]

Notice from equation 10.33 that since the distance \( r \) changes at each point in the orbit, the only way for \( L \) to remain a constant is for \( v \) to change with time. We can also use equation 10.33 to compare the speed of the planet at perihelion to its speed at aphelion. Note that in equation 10.27 \( \theta \) is the angle between the extension of \( \mathbf{r} \) and the velocity vector \( \mathbf{v} \). But at perihelion and aphelion, the angle \( \theta \) is equal to 90.0\(^\circ\). Hence the angular momentum at perihelion becomes

\[
L_p = r_p m v_p \sin 90.0^\circ = r_p m v_p
\]

and in the same way, the angular momentum at aphelion becomes

\[
L_a = r_a m v_a \sin 90.0^\circ = r_a m v_a
\]

But by the law of conservation of angular momentum, the angular momentum at perihelion must equal to the angular momentum at aphelion, therefore

\[
L_p = L_a
\]

or

\[
r_p m v_p = r_a m v_a
\]
and solving for the velocity at perihelion we get

\[ v_p = \frac{r_a}{r_p} v_a \]  

(10.34)

Equation 10.34 tells us that since the aphelion distance \( r_a \) is greater than the perihelion distance \( r_p \), the ratio \( r_a/r_p \) is greater than 1, hence the velocity \( v_p \) at perihelion is greater than the velocity \( v_a \) at aphelion.

**Example 10.9**

*Speed of the earth at perihelion.* The speed of the earth at aphelion is \( 2.92 \times 10^4 \) m/s, and the earth’s perihelion distance is approximately \( 1.47 \times 10^{11} \) m, while its aphelion distance is about \( 1.53 \times 10^{11} \) m. Determine the speed of the earth at perihelion in its orbit about the sun, shown in figure 10.18.

![Figure 10.18](image)

The speed of the earth at perihelion \( v_p \), found from equation 10.34, is

\[ v_p = \frac{r_a}{r_p} v_a = \frac{1.53 \times 10^{11} \text{ m}}{1.47 \times 10^{11} \text{ m}} \times 2.92 \times 10^4 \text{ m/s} = 3.04 \times 10^4 \text{ m/s} \]

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**Example 10.10**

*The angular momentum of the earth.* From the knowledge of the speed and distance of the earth at perihelion, determine the angular momentum of the earth. The
speed of the earth at perihelion is \(3.04 \times 10^4\) m/s, and the earth’s perihelion distance is approximately \(1.47 \times 10^{11}\) m.

**Solution**

The angular momentum of the earth is found from equation 10.33 as

\[
L = rmv \sin \theta
\]

The mass of the earth is \(5.97 \times 10^{24}\) kg and at perihelion, the angle \(\theta = 90^\circ\). Therefore

\[
L = (1.47 \times 10^{11}\) m\)(5.97 \times 10^{24}\) kg\)(3.04 \times 10^{4}\) m/s \sin 90^\circ
\]

\[
= 2.67 \times 10^{40}\) kg m^2/s
\]

**The Speed of a Planet in an Elliptical Orbit**

Just as the speed of a planet in a circular orbit is determined by equation 10.26, the speed of a planet in an elliptical orbit can also be determined. However, the mathematical derivation is somewhat more complicated and will not be given here, but the result is quite simple. *The speed of a planet in an elliptical orbit is given by*

\[
v = \sqrt{\frac{GM}{r} \left( \frac{2}{r} - \frac{1}{a} \right)}
\]  

(10.35)

where \(r\) is the position of the planet in the orbit and \(a\) is the semimajor axis of the ellipse. For the special case of a circular orbit \(a = r\), and equation 10.35 becomes

\[
v = \sqrt{\frac{GM}{r} \left( \frac{2}{r} - \frac{1}{r} \right)}
\]

\[
v = \sqrt{\frac{GM}{r}}
\]

Notice that this is the same equation for the speed of a planet in a circular orbit, that we developed in equation 10.26, and we see that the circular orbit is only a special case of the elliptical orbit.

**Example 10.11**

The speed of the earth at its perihelion and aphelion position. The earth is at its perihelion position on about January 3 when it is approximately \(1.47 \times 10^{11}\) m away from the sun. The earth reaches its aphelion distance, on July 4, when it is
about $1.53 \times 10^{11}$ m away from the sun. Find the speed of the earth at its perihelion and aphelion position in its orbit.

**Solution**

The semimajor axis of the elliptical orbit, found from equation 10.22, is

$$a = \frac{r_a + r_p}{2} = \frac{1.53 \times 10^{11} \text{ m} + 1.47 \times 10^{11} \text{ m}}{2} = 1.50 \times 10^{11} \text{ m}$$

The speed of the earth at perihelion is found from equation 10.35 with $r = r_p$, the perihelion distance,

$$v = \sqrt{GM_a \left(\frac{2}{r} - \frac{1}{a}\right)}$$

Hence,

$$v_p = \sqrt{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)(1.99 \times 10^{30} \text{ kg})(2/1.47 \times 10^{11} \text{ m} - 1/1.50 \times 10^{11} \text{ m})} = 3.03 \times 10^4 \text{ m/s}$$

The speed of the earth at aphelion is found from equation 10.35 with $r = r_a = 1.53 \times 10^{11}$ m,

$$v_a = \sqrt{GM_a \left(\frac{2}{r_a} - \frac{1}{a}\right)}$$

$$v_a = \sqrt{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)(1.99 \times 10^{30} \text{ kg})(2/1.53 \times 10^{11} \text{ m} - 1/1.50 \times 10^{11} \text{ m})} = 2.92 \times 10^4 \text{ m/s}$$

The speed of the earth at any position $r$ in its elliptical orbit can thus be found from equation 10.35. It is easy to see why the earth, in its orbit about the sun, is sometimes approximated as a circular orbit. The aphelion distance, perihelion distance, and the mean distance are very close, that is, $1.53 \times 10^{11}$ m, $1.47 \times 10^{11}$ m, and $1.50 \times 10^{11}$ m, respectively. Also the speed of the earth at aphelion, perihelion, and in a circular orbit is $2.92 \times 10^4$ m/s, $3.03 \times 10^4$ m/s, and $2.97 \times 10^4$ m/s, respectively, which are also very close. The error in using the circular approximation rather than the elliptical analysis is no more than about 2%.
Kepler’s Third law of Planetary Motion

Kepler’s third law of planetary motion says that the square of the time for the planet to make a complete revolution about the sun is proportional to the cube of the semimajor axes of the elliptical orbit. The time for the planet to make a complete revolution about the sun is called its period. To simplify our discussion we consider the planet to be in a circular orbit. The speed of the planet in a circular orbit can be found as the total distance the planet moves divided by the time $T$ for the planet to move that distance. The total distance the planet moves is just the circumference of the circle, which is $2\pi r$, hence

$$v = \frac{2\pi r}{T}$$

But we already found that the speed of a planet in a circular orbit was given by equation 10.26 as

$$v = \sqrt{\frac{Gms}{r}}$$

Equating the two of these we get

$$v = \sqrt{\frac{Gms}{r}} = \frac{2\pi r}{T}$$

and solving for $T$

$$T = \frac{2\pi r}{\sqrt{\frac{Gms}{r}}}$$

Squaring both sides of the equation we get

$$T^2 = \frac{4\pi^2 r^2}{Gms}$$

or

$$T^2 = \left(\frac{4\pi^2}{Gms}\right)r^3$$

(10.36)

Equation 10.37 is a statement of Kepler’s third law of planetary motion and as you can see the square of the period of motion of the planet is proportional to the cube of the radius of the circular orbit. The actual period is found by taking the square root of equation 10.36 and is

$$T = \sqrt{\left(\frac{4\pi^2}{Gms}\right)r^3}$$

(10.37)

---

2\(^2\)Recall that the semimajor axis of an ellipse becomes the radius when the ellipse degenerates into a circle.
Example 10.12

*The period of motion of a planet.* Find the period of motion of the earth in its orbit about the sun.

**Solution**

The mean radius of the earth in a circular orbit is approximately $1.50 \times 10^{11}$ m. The period of the motion is found from equation 10.37 as

$$T = \sqrt[3]{\frac{4\pi^2}{GM_s}} r^3$$

$$T = \sqrt[3]{\frac{4\pi^2}{(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)(1.99 \times 10^{30} \text{ kg})}(1.50 \times 10^{11} \text{ m})^3}$$

$$= 3.17 \times 10^7 \text{ s} = 1.00 \text{ yr.}$$

**To go to this Interactive Example click on this sentence.**

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The Earth and Its Satellite
10.7 Satellite Motion

The analysis of satellite motion is essentially the same as planetary motion. Although in planetary motion, the planets are all in elliptical orbits, a satellite can be in a circular, elliptical, parabolic or hyperbolic orbit. Let us consider the motion of a satellite around its parent body. This could be the motion of the moon around the earth, the motion of any other moon around its planet, or the motion of an artificial satellite around the earth, the moon, another planet, and so forth.

Let us start with the analysis of the motion of an artificial satellite in a circular orbit around the earth. Perhaps the first person to ever conceive of the possibility of an artificial earth satellite was Sir Isaac Newton, when he wrote in 1686 in his *Principia*:

> But if we now imagine bodies to be projected in the directions of lines parallel to the horizon from greater heights, as of 5, 10, 100, 1000, or more miles or rather as many semi-diameters of the earth, those bodies according to their different velocity, and the different force of gravity in different heights, will describe arcs either concentric with the earth, or variously eccentric, and go on revolving through the heavens in those orbits just as the planets do in their orbits.3

Let us consider the motion of a satellite in a circular motion. For the satellite to be in motion in a circular orbit, there must be a centripetal force acting on the satellite to force it into the circular motion. This centripetal force acting on the satellite, is supplied by the force of gravity of the earth. Let us assume that the satellite is in orbit a distance $h$ above the surface of the earth, as shown in figure 10.19. Because the centripetal force is supplied by the gravitational force, we have

$$ F_c = F_g $$

or

$$ \frac{m_sv^2}{r} = \frac{Gm_em_s}{r^2} $$

Solving for the speed of the satellite in the circular orbit, we obtain

---

The first thing that we note is that $m_s$, the mass of the satellite, divided out of equation 10.38. This means that the speed of the satellite is independent of its mass. That is, the speed is the same, whether it is a very large massive satellite or a very small one.

Equation 10.39 represents the speed that a satellite must have if it is to remain in a circular orbit, at a distance $r$ from the center of the earth. Because the satellite is at a height $h$ above the surface of the earth, the orbital radius $r$ is

\[ r = r_e + h \]  

(10.40)

Equation 10.39 also says that the speed depends only on the radius of the orbit $r$. For large values of $r$, the required speed will be relatively small; whereas for small values of $r$, the speed $v$ must be much larger. If the actual speed of a satellite at a distance $r$ is less than the value of $v$, given by equation 10.39, then the satellite will move closer toward the earth. If it gets close enough to the earth’s atmosphere, the air friction will slow the satellite down even further, eventually causing it to spiral into the earth. The increased air friction will then cause it to burn up and crash. If the actual speed at the distance $r$ is greater than that given by equation 10.39, then the satellite will go farther out into space, eventually going into either an elliptical, parabolic, or hyperbolic orbit, depending on the speed $v$.

**Example 10.13**

*Speed of a satellite in a circular orbit.* A satellite orbits the earth 1500 km above the surface. Find its speed and its period of revolution.

**Solution**

The mass of the earth is $m_e = 5.97 \times 10^{24}$ kg, and its mean radius, found in table 10.2, is $r_e = 6.37 \times 10^6$ m. The orbital radius of the satellite is found from equation 10.40 as

\[ r = r_e + h \]

\[ r = 6.37 \times 10^6 \text{ m} + 1.50 \times 10^6 \text{ m} \]

\[ r = 7.87 \times 10^6 \text{ m} \]

The speed of the satellite around the earth, found from equation 10.39, is

\[ v = \sqrt{\frac{Gm_e}{r}} \]
= \sqrt{\left(\frac{6.67 \times 10^{-11} \text{N} \cdot \text{m}^2}{\text{kg}^2}\right) \frac{5.97 \times 10^{24} \text{kg}}{7.87 \times 10^6 \text{m}}} \\
= 7.11 \times 10^3 \text{ m/s}

Thus the speed of the satellite in its circular orbit about the earth is $7.11 \times 10^3 \text{ m/s}$.

The period of revolution of the satellite is found in the same way that the period of a planet about the sun was found, and we can use equation 10.37, with the mass of the sun replaced with the mass of the earth. That is, the period is

$$T = \sqrt{\frac{4\pi^2}{Gm_e}r^3}$$

$$T = \sqrt{\left(\frac{4\pi^2}{(6.67 \times 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2)(5.97 \times 10^{24} \text{ kg})}\right)(7.87 \times 10^6 \text{ m})^3}$$

$$= 6.95 \times 10^3 \text{ s} = 116 \text{ min.} = 1.93 \text{ hr.}$$

That is, if the satellite is directly above you now, it will take almost 2 hours before it is directly above you again.

To go to this Interactive Example click on this sentence.

We have found the speed that is necessary to place a satellite into orbit, but how do we get the satellite into this circular orbit? The satellite is placed in the orbit by a rocket. The rocket is launched vertically from the earth as described in chapter 8, and at a predetermined altitude it pitches over, so as to approach the desired circular orbit tangentially. The engines are usually turned off and the rocket coasts on a projectile trajectory to the orbital intercept point $I$ in figure 10.20.

![Figure 10.20 Placing a satellite in a circular orbit.](image)

Let the velocity of the rocket on the ascent trajectory at the point of intercept be $v_A$. The velocity necessary for the satellite to be in circular orbit at the height $h$ is $v$ and
Chapter 10 Gravitation - Planetary and Satellite Motion

its speed is given by equation 10.39. Therefore, the rocket must undergo a change in velocity $\Delta v$ to match its ascent velocity to the velocity necessary for the circular orbit. That is,

$$\Delta v = v - v_A$$  \hspace{1cm} (10.41)

This change in velocity is of course produced by the thrust of the rocket engines. How long should these engines be turned on to get this necessary change in speed $\Delta v$? As a first approximation we take Newton's second law in the form

$$F = ma = m\frac{\Delta v}{\Delta t}$$

Solving Newton's second law for $\Delta t$, gives

$$\Delta t = \frac{m \Delta v}{F}$$  \hspace{1cm} (10.42)

where $F$ is the thrust of the rocket engine, $\Delta v$ is the necessary speed change, determined from equation 10.41; and $m$ is the mass of the spacecraft at this instant of time. Therefore, equation 10.42 tells the astronaut the length of time to “burn” his engines. At the end of this time the engines are shut off, and the spacecraft has the necessary orbital speed to stay in its circular orbit.

This is, of course, a greatly simplified version of orbital insertion, for we need not only the magnitude of $\Delta v$ but also its direction. An attitude control system is necessary to determine the proper direction for the $\Delta v$. Also it is important to note that using equation 10.42 is only an approximation, because as the spacecraft burns its rocket propellant, its mass $m$ is continually changing. This example points out a deficiency in using Newton's second law in the form $F = ma$, because this form assumes that the mass under consideration is a constant.

The Geosynchronous Satellite

An interesting example of satellite motion is the geosynchronous satellite. The geosynchronous satellite is a satellite whose orbital motion is synchronized with the rotation of the earth. In this way the geosynchronous satellite is always over the same point on the equator as the earth turns, figure 10.21(a). The geosynchronous satellite is obviously very useful for world communication, weather observations, and military use.

What should the orbital radius of such a satellite be, in order to stay over the same point on the earth’s surface? The speed necessary for the circular orbit, given by equation 10.39, is

$$v = \sqrt{\frac{Gm_e}{r}}$$

But this speed must be equal to the average speed of the satellite in one day, namely
where $T$ is the period of revolution of the satellite that is equal to one day. That is, the satellite must move in one complete orbit in a time of exactly one day. Because the earth rotates in one day and the satellite will revolve around the earth in one day, the satellite at $A'$ will always stay over the same point on the earth $A$, as in figure 10.21(a). That is, the satellite is at $A'$, which is directly above the point $A$ on the earth. As the earth rotates, $A'$ is always directly above $A$. Setting equation 10.43 equal to equation 10.39 for the speed of the satellite, we have

$$\frac{2\pi r}{T} = \sqrt{\frac{Gm_e}{r}}$$

Squaring both sides of equation 10.44 gives

$$\frac{4\pi^2 r^2}{T^2} = \frac{Gm_e}{r}$$

or

$$r^3 = \frac{Gm_e T^2}{4\pi^2}$$

Solving for $r$, gives the required orbital radius of

$$r = \left(\frac{Gm_e T^2}{4\pi^2}\right)^{1/3}$$

Substituting the values for the earth into equation 10.45 gives
the orbital radius, measured from the center of the earth, for a geosynchronous satellite. A satellite at this height will always stay directly above a particular point on the surface of the earth.

A satellite communication system can be set up by placing several geosynchronous satellites in orbits over different points on the surface of the earth. As an example, suppose four geosynchronous satellites were placed in orbit, as shown in figure 10.21(b). Let us say that we want to communicate, by radio or television, between the points A and B, which are on opposite sides of the earth. The communication would first be sent from point A to geosynchronous satellite 1, which would retransmit the communication to geosynchronous satellite 2. This satellite would then transmit to geosynchronous satellite 3, which would then transmit to the point B on the opposite side of the earth. Since these geosynchronous satellites appear to hover over one place on earth, continuous communication with any place on the surface of the earth can be attained.

10.8 Space Travel.

Earth is the cradle of man, but man was never made to stay in a cradle forever.  
K. Tsiolkovsky

Have you ever wondered what it would be like to go to the moon or perhaps to another planet or to travel anywhere in outer space? But how can you get there? How can you travel into space?

Man has long had a fascination with the possibility of space travel. Jules Verne’s novel, From the Earth to the Moon, was first published in 1868. In it he describes a trip to the moon inside a gigantic cannon shell. It is interesting to note that he says

Now as the Moon is never in the zenith, or directly overhead, in countries further than 28° from the equator, to decide on the exact spot for casting the Columbiad became a question that required some nice consultation. [And then a little further on] The 28th parallel of north latitude, as every schoolboy knows, strikes the American continent a little below Cape Canaveral. (pp. 66 and 68)

As I am sure we all know, Cape Canaveral is the site of the present Kennedy Space Center, the launch site for the Apollo mission to the moon. The first astronauts landed on the moon on July 20, 1969, just over a hundred years after the publication of Verne’s novel. (Actually Jules Verne did not pick Cape Canaveral
The idea of space travel left the realm of science fiction by the work of three men, Konstantin Tsiolkovsky, a Russian; Robert Goddard, an American; and Hermann Oberth, a German. Tsiolkovsky’s first paper, “Free Space,” was published in 1883. In his Dreams of Earth and Sky, 1895, he wrote of an artificial earth satellite. Goddard’s first paper, “A Method of Reaching Extreme Altitudes,” was written in 1919. The extreme altitudes he was referring to was the moon. Goddard launched the first liquid-fueled rocket in history on March 16, 1926. Meanwhile, Oberth published his work, The Rocket into Inter Planetary Space, in 1923, which culminated with the German V-2 rocket in World War II. Another analysis of the problems associated with space flight was published in 1925 by Walter Hohmann in Die Erreichbarkeit der Himmelskorper (The Attainability of the Heavenly Bodies). In the preface, Hohmann says,

_The present work will contribute to the recognition that space travel is to be taken seriously and that the final successful solution of the problem cannot be doubted, if existing technical possibilities are purposefully perfected as shown by conservative mathematical treatment._

Hohmann’s original work had been written 10 years previous to its publication. In this work, Hohmann shows how to get to the Moon, Venus, and Mars. His simple approach to reach these heavenly bodies is by use of the cotangential ellipse.

The simplest approach to space flight to the moon or to a planet is by use of the Hohmann transfer ellipse. Let us assume that the spacecraft is launched from the surface of the earth on an ascent trajectory. It is then desired to place the spacecraft in a circular parking orbit about the earth. If the circular parking orbit is to be at a height $h_e$ above the surface of the earth then the necessary speed for the spacecraft, given by equation 10.39, is

$$v_{oe} = \sqrt{\frac{GM_e}{r_e + h_e}}$$

(10.46)

Knowing the speed of the spacecraft on the ascent trajectory from an on-board inertial navigational system, equation 10.41 is then used to determine the necessary “delta $v$,” $\Delta v$, to get into this orbit. The engines are then turned on for the value of $\Delta t$, determined by equation 10.42, and the spacecraft is thus inserted into the circular parking orbit about the earth.

Before descending to the surface of the moon, it would be desirable to first go into a circular lunar parking orbit. To get to this circular lunar parking orbit, a “cotangential ellipse,” the Hohmann transfer ellipse, is placed onto the two parking

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44 From the Earth to the Moon in The Space Novels of Jules Verne, p. 69, Dover Publications, N.Y.
orbits, such that the focus of the ellipse is placed at the center of mass of the earth-moon system, and the ellipse is tangential to each parking orbit, as seen in figure 10.22. All positions in the orbit are measured from the center of mass of the earth-moon system. The semimajor axis $a$, of this ellipse, found from figure 10.22, is

$$a = \frac{r_{em} + r_m + h_m + r_e + h_e}{2}$$

(10.47)

where

- $r_{em}$ is the distance from the center of the earth to the center of the moon.
- $r_m$ is the radius of the moon.
- $h_m$ is the height of the spacecraft above the surface of the moon.
- $r_e$ is the radius of the earth.
- $h_e$ is the height of the spacecraft above the surface of the earth when it is in its circular parking orbit.

The insertion of the spacecraft into the transfer ellipse occurs at the perigee position of the elliptical orbit, which from figure 10.22 is

$$r_p = r_{cm} + r_e + h_e$$

(10.48)

where $r_{cm}$ is the distance from the center of earth to the center of mass of the earth-moon system. The necessary speed to get into this cotangential ellipse at the perigee position, found from equation 10.35, is

$$v_{TEp} = \sqrt{\frac{GM_e}{r_p} \left(\frac{2}{r_p} - \frac{1}{a}\right)}$$

(10.49)

The word perigee is used to denote the point in the orbit where the satellite is closest to the earth. The word perigee is to an earth orbit what perihelion is to a solar orbit.
where \( a \) and \( r_p \) are found from equations 10.47 and 10.48, respectively. The notation \( v_{TEp} \) stands for the speed in the transfer ellipse at perigee.

Because the speed of the spacecraft in the earth parking orbit is known, equation 10.46, and the necessary speed for the transfer orbit is known, equation 10.49, the necessary change in speed (\( \Delta v \)) of the spacecraft is just the difference between these speeds. Hence, the required \( \Delta v \) for insertion into the transfer ellipse is given by

\[
\Delta v_1 = v_{TEp} - v_{oe}
\]  

(10.50)

The spacecraft engines must be turned on to supply this necessary change in speed (\( \Delta v \)). When this \( \Delta v_1 \) is achieved, the spacecraft engines are turned off and the spacecraft coasts toward the moon. If the engines are not turned on again, then the spacecraft would coast to the moon, reach it, and would then continue back toward the earth on the second half of the transfer ellipse. Thus, if there were some type of malfunction on the spacecraft, it would automatically return to earth.

Assuming there is no failure, the astronauts on board the spacecraft would like to change from their transfer orbit into the circular lunar parking orbit. The speed of the spacecraft on the transfer ellipse is given by equation 10.35, with \( r = r_a \) the apogee distance, as

\[
v_{TEa} = \sqrt{\frac{G M_e}{2 - \frac{1}{a}}} \tag{10.51}
\]

The apogee distance \( r_a \), found from figure 10.22, is

\[
r_a = r_{em} + r_m + h_m - r_{cm}
\]  

(10.52)

The necessary speed that the spacecraft must have to enter a circular lunar parking orbit \( v_{om} \) is found from modifying equation 10.39 to

\[
v_{om} = \sqrt{\frac{G M_m}{r_m + h_m}}
\]  

(10.53)

where \( M_m \) is the mass of the moon, \( r_m \) is the radius of the moon, and \( h_m \) is the height of the spacecraft above the surface of the moon in its circular lunar parking orbit. The necessary change in speed to transfer from the Hohmann ellipse to the circular lunar parking orbit is obtained by subtracting equation 10.53 from equation 10.51. Thus the necessary \( \Delta v \) is

\[
\Delta v_{II} = v_{TEa} - v_{om}
\]  

(10.54)

The spacecraft engines are turned on to obtain this necessary change in speed. When the engines are shut off the spacecraft will have the speed \( v_{om} \), and will stay
in the circular lunar parking orbit until the astronauts are ready to descend to the lunar surface. The process is repeated for the return to earth.

The Hohmann transfer is the simplest of the transfer orbits and is also the orbit of minimum energy. However, it has the disadvantage of having a large flight time. In the very early stages of the Apollo program, the Hohmann transfer ellipse was considered for the lunar transfer orbit. However, because of its long flight time, it was discarded for a hyperbolic transfer orbit that had been perfected by the Jet Propulsion Laboratories in California on its Ranger, Surveyor, and Lunar Orbiter unmanned spacecrafts to the moon. The hyperbolic orbit requires a great deal more energy, but its flight time is relatively small. The procedure for a trip on a hyperbolic orbit is similar to the elliptical orbit, only another equation is necessary for the speed of the spacecraft in the hyperbolic orbit. The principle however is the same. Determine the current speed in the particular orbit, then determine the speed that is necessary for the other orbit. The difference between the two of them is the necessary $\Delta v$. The spacecraft engines are turned on until this value of $\Delta v$ is obtained. A typical orbital picture for this type of transfer is shown in figure 10.23.

Unmanned satellites have since traveled to Mars, Venus, Saturn, Jupiter, Uranus, and Neptune. At 1:07 PM (EDT) on July 4, 1997 the Mars Pathfinder spacecraft set down on the surface of Mars. Although unmanned missions to Mars have already occurred, the Pathfinder mission was unique because for the first time it carried a robotic land rover. The rover, called Sojourner, figure 10.24, which can be controlled from earth, moved out and explored the Martian landscape, sending back beautiful pictures of that landscape and other scientific data.
And what about manned trips to these planets? On July 20, 1989, the twentieth anniversary of the first landing on the moon, the president of the United States, George Bush, announced to the world that the United States will begin planning a manned trip to the planet Mars and eventually to an exploration of our entire solar system. Although no manned trip has yet occurred, it is safe to assume that man is getting ready to leave his cradle.

**The Language of Physics**

**Newton’s law of universal gravitation**
Between every two masses in the universe there is a force of attraction that is directly proportional to the product of their masses and inversely proportional to the square of the distance separating them (p. ).

**Kepler’s first law of planetary motion**
The orbit of each planet is an ellipse with the sun at one focus. We can sometimes approximate the orbit as if it were circular. (p. ).

**Kepler’s second law of planetary motion**
The speed of the planet varies in such a way that the line joining the planet and the sun sweeps out equal areas in equal times. The most important consequence of these equal areas in equal times is that the speed of the planet is not a constant in
an elliptical orbit as it was in a circular orbit. Kepler's second law is a consequence of the law of conservation of angular momentum applied to the planet in its orbit about the sun. (p. ).

**Kepler’s third law of planetary motion**
The square of the time for the planet to make a complete revolution about the sun is proportional to the cube of the semimajor axes of the elliptical orbit (p. ).

**The velocity of escape**
The velocity of escape for a rocket is the minimum velocity that a rocket must have to leave the surface of the earth and not be pulled back by the gravitational force of the earth. (p. ).

**Satellite Motion**
Satellite motion is essentially the same as planetary motion. However, a satellite can be in a circular, elliptical, parabolic or hyperbolic orbit. The satellite motion could be the motion of the moon around the earth, the motion of any other moon around its planet, or the motion of an artificial satellite around the earth, the moon, another planet, and so forth. (p. ).

**Geosynchronous satellite**
A satellite whose orbital motion is synchronized with the rotation of the earth. In this way the satellite is always over the same point on the equator as the earth turns (p. ).

**Hohmann transfer ellipse**
Hohmann proposed the transfer ellipse as a simple way to get to the Moon, Venus, and Mars. For the moon, the “cotangential ellipse,” is placed onto the two parking orbits, such that the focus of the ellipse is placed at the center of mass of the earth-moon system, and the ellipse is tangential to each parking orbit, one around the earth, the other around the moon.

### Summary of Important Equations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = \frac{G m_1 m_2}{r^2}$</td>
<td>Newton’s law of universal gravitation</td>
</tr>
<tr>
<td>$F_{21} = -G \frac{m_1 m_2}{r^2} r_o$</td>
<td>(10.4)</td>
</tr>
<tr>
<td>$g_e = \frac{G m_e}{r^2}$</td>
<td>The acceleration due to gravity on earth</td>
</tr>
<tr>
<td>$g_{\phi} = \frac{G m_e}{r^2}$</td>
<td>The acceleration due to gravity at a particular latitude $\phi$ taking the rotation of the earth into account.</td>
</tr>
</tbody>
</table>
where the centripetal acceleration is a function of the latitude angle $\phi$ and is given by

$$a_c = \omega^2 r = \omega^2 R \cos \phi \quad (10.10)$$

The acceleration due to gravity on the moon $g_m = \frac{Gm_m}{r_m^2} \quad (10.11)$

The difference in potential energy between the point A and the point B in the gravitational field of a mass $m_1$.

$$PE_B - PE_A = -G m_1 m_2 \left( \frac{1}{r_B} - \frac{1}{r_A} \right) \quad (10.18)$$

The potential energy of a mass $m_1$ a distance $r$ from the mass $m_2$.

$$PE = -\frac{G m_1 m_2}{r} \quad (10.19)$$

The magnitude of the velocity of escape for a rocket to leave the surface of the earth and not be pulled back by the gravitational force of the earth.

$$v_e = \sqrt{\frac{2GM}{R}} \quad (10.21)$$

The semimajor axis of an ellipse,

$$a = \frac{r_a + r_p}{2} \quad (10.22)$$

The eccentricity is a measure of the ellipticity of the orbit

$$e = \frac{r_a - r_p}{2a} \quad (10.23)$$

Speed of a planet in a circular orbit

$$v = \sqrt{\frac{G m_s}{r}} \quad (10.26)$$

Kepler’s second law of planetary motion, the rate at which the area of an orbit changes with time is equal to a constant. This is a consequence of the law of conservation of angular momentum

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant} \quad (10.32)$$

The speed of a planet in an elliptical orbit

$$v = \sqrt{GM \left( \frac{2}{r} - \frac{1}{a} \right)} \quad (10.35)$$

Kepler’s third law of planetary motion for a circular orbit

$$T^2 = \left( \frac{4\pi^2}{G m_s} \right) r^3 \quad (10.36)$$
Chapter 10  Gravitation - Planetary and Satellite Motion

The period of motion for a circular orbit

\[ T = \sqrt{\left(\frac{4\pi^2}{Gm_s}\right)r^3} \]  

(10.37)

Speed of a satellite in a circular orbit

\[ v = \sqrt{\frac{Gm_e}{r}} \]  

(10.39)

The orbital radius \( r \) is

\[ r = r_e + h \]  

(10.40)

Rocket engine “burn” time

\[ \Delta t = \frac{m \Delta v}{F} \]  

(10.42)

Radius for geosynchronous orbit

\[ r = \left(\frac{Gm_e T^2}{4\pi^2}\right)^{1/3} \]  

(10.45)

Questions for Chapter 10

1. If the acceleration of gravity varies from place to place on the surface of the earth, how does this affect records made in the Olympics in such sports as shot put, javelin throwing, high jump, and the like?

2. What is wrong with applying Newton’s second law in the form \( F = ma \) to satellite motion? Does this same problem occur in the motion of an airplane?

*3. How can you use Kepler’s second law to explain that the earth moves faster in its motion about the sun when it is closer to the sun?

4. Could you place a synchronous satellite in a polar orbit about the earth? At 45\(^\circ\) latitude?

5. Explain how you can use a Hohmann transfer orbit to allow one satellite in an earth orbit to rendezvous with another satellite in a different earth orbit.

*6. A satellite is in a circular orbit. Explain what happens to the orbit if the engines are momentarily turned on to exert a thrust (a) in the direction of the velocity, (b) opposite to the velocity, (c) toward the earth, and (d) away from the earth.

7. A projectile fired close to the earth falls toward the earth and eventually crashes to the earth. The moon in its orbit about the earth is also falling toward the earth. Why doesn’t it crash into the earth?

*8. The gravitational force on the earth caused by the sun is greater than the gravitational force on the earth caused by the moon. Why then does the moon have a greater effect on the tides than the sun?

*9. How was the universal gravitational constant \( G \) determined experimentally?
Problems for Chapter 10

10.1 Newton’s Law of Universal Gravitation

1. Two large metal spheres are separated by a distance of 2.00 m from center to center. If each sphere has a mass of 5000 kg, what is the gravitational force between them?

2. A 5.00-kg mass is 1.00 m from a 10.0-kg mass. (a) What is the gravitational force that the 5.00-kg mass exerts on the 10.0-kg mass? (b) What is the gravitational force that the 10.0-kg mass exerts on the 5.00-kg mass? (c) If both masses are free to move, what will their initial acceleration be?

3. Three point masses of 10.0 kg, 20.0 kg, and 30.0 kg are located on a line at 10.0 cm, 50.0 cm, and 80.0 cm, respectively. Find the resultant gravitational force on (a) the 10.0-kg mass, (b) the 20.0-kg mass, and (c) the 30.0-kg mass.

4. A boy meets a girl for the first time and is immediately attracted to her. If he has a mass of 75.0 kg and she has a mass of 50.0 kg and they are separated by a distance of 3.00 m, is their attraction purely physical?

5. What is the gravitational force between a proton and an electron in a hydrogen atom if they are separated by a distance of $5.29 \times 10^{-11}$ m?

10.2-10.4 The Acceleration Due to Gravity

6. What is the value of $g$ at a distance from the center of the earth of (a) 1 earth radius, (b) 2 earth radii, (c) 10 earth radii, and (d) at the distance of the moon?

7. What is the weight of a body, in terms of its weight at the surface of the earth, at a distance from the center of the earth of (a) 1 earth radius, (b) 2 earth radii, (c) 10 earth radii, and (d) at the distance of the moon? How can an object in a satellite, at say 2 earth radii, be considered to be weightless?

8. Taking the earth’s rotation into account, determine the acceleration due to gravity at (a) 30.0°N latitude, and (b) 60.0°N latitude

9. Calculate the acceleration of gravity on the surface of Mars. What would a man who weighs 801 N on earth weigh on Mars?

*10. It is the year 2020 and a base has been established on Mars. An enterprising businessman decides to buy coffee on earth at $1.12/N and sell it on
Mars for $2.25/N. How much does he make or lose per newton when he sells it on Mars? Ignore the cost of transportation from earth to Mars.

11. The sun’s radius is 110 times that of the earth, and its mass is 333,000 times as large. What would be the weight of a 1.00-kg object at the surface of the sun, assuming that it does not melt or evaporate there?

10.5 Gravitational Potential Energy

12. Assuming that the orbit of the earth about the sun is a circle of radius \( r = 1.50 \times 10^{11} \) m, find the potential energy of the sun-earth system.

13. Find the difference in potential energy of a satellite of mass \( m_s = 2.50 \times 10^3 \) kg when it is at a distance of \( 2.55 \times 10^9 \) m and when it is at the distance of \( 4.30 \times 10^7 \) m from the center of the earth.

14. Find the velocity necessary for a rocket to escape from the gravitational field of the planet (a) Mars and (b) Jupiter.

10.6 Planetary Motion

15. What is the velocity of the moon around the earth in a circular orbit? What is the time for one revolution?

16. Calculate the velocity in an approximate circular orbit about the sun, and the time for one revolution of the planet (a) Mars and (b) Jupiter.

17. Show that if the semimajor axis \( a \) and the eccentricity \( e \) of an elliptical orbit are known, then the aphelion and perihelion distances are given by

\[
\begin{align*}
    r_a &= a(1 + e) \\
    r_p &= a(1 - e)
\end{align*}
\]

18. The semimajor axis of the planet Mars is \( 2.28 \times 10^{11} \) m and its eccentricity is 0.093. Using the results of problem 17, find the aphelion and perihelion distances of the planet Mars.

19. Using the results of problem 18, find the velocity of Mars at perihelion and at aphelion in its orbit about the sun.

10.7 Satellite Motion

20. A satellite is in a circular orbit 1130 km above the surface of the earth. Find its speed and its period of revolution.

21. Calculate the speed of a satellite orbiting 100 km above the surface of Mars. What is its period?

*22. An Apollo space capsule orbited the moon in a circular orbit at a height of 112 km above the surface. The time for one complete orbit, the period \( T \), was 120 min. Find the mass of the moon.

*23. A satellite orbits the earth in a circular orbit in 130 min. What is the distance of the satellite to the center of the earth? What is its height above the surface? What is its speed?
24. Apollo 10 first went into an elliptical orbit about the moon that was at a maximum distance of $3.15 \times 10^5$ m to a low of $1.11 \times 10^5$ m above the lunar surface. Find the semimajor axis of this elliptical orbit and its eccentricity.

10.8 Space Travel

*25. If a spacecraft is to transfer from a 370 km earth parking orbit to a 150 km lunar parking orbit by a Hohmann transfer ellipse, find (a) the location of the center of mass of the earth-moon system, (b) the perigee distance of the transfer ellipse, (c) the apogee distance, (d) the semimajor axis of the ellipse, (e) the speed of the spacecraft in the earth circular parking orbit, (f) the speed necessary for insertion into the Hohmann transfer ellipse, (g) the necessary $\Delta v$ for this insertion, (h) the speed of the spacecraft in a circular lunar parking orbit, (i) the speed of the spacecraft on the Hohmann transfer at time of lunar insertion, and (j) the necessary $\Delta v$ for insertion into the lunar parking orbit.

Additional Problems

26. Find the resultant vector acceleration caused by the acceleration due to gravity and the centripetal acceleration for a person located at (a) the equator, (b) 45.0° north latitude, and (c) the north pole.

*27. Three point masses of 30.0 kg, 50.0 kg, and 70.0 kg are located at the vertices of an equilateral triangle 1.00 m on a side. Find the resultant gravitational force on each mass.

*28. Four metal spheres are located at the corners of a square of sides of 0.300 m. If each sphere has a mass of 10.0 kg, find the force on the sphere in the lower right-hand corner.

29. What is the gravitational force between the earth and the moon? If a steel cable can withstand a force of $7.50 \times 10^4$ N/cm², what must the diameter of a steel cable be to sustain the equivalent force?

*30. At what speed would the earth have to rotate such that the centripetal force at the equator would be equal to the weight of a body there? If the earth rotated at this velocity, how long would a day be? If a 890-N man stood on a weighing scale there, what would the scales read?

*31. What would the mass of the earth have to be in order that the gravitational force is inadequate to supply the necessary centripetal force to keep a person on the surface of the earth at the equator? What density would this correspond to? Compare this to the actual density of the earth.

*32. Compute the gravitational force of the sun on the earth. Then compute the gravitational force of the moon on the earth. Which do you think would have a greater effect on the tides, the sun or the moon? Which has the greatest effect?

*33. Find the force exerted on 1.00 kg of water by the moon when (a) the 1.00 kg is on the side nearest the moon and (b) when the 1.00 kg is on the side farthest from the moon. Would this account for tides?

*34. By how much does (a) the sun and (b) the moon change the value of $g$ at the surface of the earth?
35. How much greater would the range of a projectile be on the moon than on the earth?

*36. Find the point between the earth and the moon where the gravitational forces of earth and moon are equal. Would this be a good place to put a satellite?

*37. An earth satellite is in a circular orbit 177 km above the earth. The period, the time for one orbit, is 88.0 min. Determine the velocity of the satellite and the acceleration of gravity in the satellite at the satellite altitude.

*38. Using Kepler’s third law, find the mass of the sun. If the radius of the sun is \( 7.00 \times 10^8 \) m, find its density.

*39. The speed of the earth around the sun was found, using dynamical principles in the example 10.8 of section 10.6, to be 29.7 km/s. Show that this result is consistent with a purely kinematical calculation of the speed of the earth about the sun.

*40. A better approximation for equation 10.42, the “burn time” for the rocket engines, can be obtained if the rate at which the rocket fuel burns, is a known constant. The rate at which the fuel burns is then given by

\[
\frac{\Delta m}{\Delta t} = K.
\]

Hence, the mass at any time during the burn will be given by \( m_0 - K \Delta t \), where \( m_0 \) is the initial mass of the rocket ship before the engines are turned on. Show that for this approximation the time of burn becomes

\[
\Delta t = \frac{m_0 \Delta v}{F + K \Delta v}
\]

41. Find the ratio of the escape velocity of a rocket to the velocity of the rocket in a circular orbit.

42. Find the total energy of a satellite in a circular orbit about the earth. What does the fact that the energy is negative, imply?

**Interactive Tutorials**

43. **Newton’s law of gravity.** Two masses \( m_1 = 5.10 \times 10^{21} \) kg and \( m_2 = 3.00 \times 10^{14} \) kg are separated by a distance \( r = 4.30 \times 10^5 \) m. Calculate their gravitational force of attraction.

44. **Acceleration due to gravity.** Planet X has mass \( m_p = 3.10 \times 10^{25} \) kg and a radius \( r_p = 5.40 \times 10^7 \) m. Calculate the acceleration due to gravity \( g \) at distances of 1-10 planet radii from the planet’s surface.

45. **Speed of a satellite.** Find the speed of a satellite in a circular orbit about its parent body.

46. **Space flight.** You are to plan a trip to the planet Mars using the Hohmann transfer ellipse described in section 10.8. The spacecraft is to transfer from a 925-km earth circular parking orbit to a 185-km circular parking orbit around Mars. Find (a) the center of mass of the Earth-Sun-Mars system, (b) the
perigee distance of the transfer ellipse, (c) the apogee distance of the transfer
ellipse, (d) the semimajor axis of the ellipse, (e) the speed of the spacecraft in the
earth parking orbit, (f) the speed necessary for insertion into the Hohmann transfer
ellipse, (g) the necessary $\Delta v$ for insertion into the transfer ellipse, (h) the necessary
speed in the Mars circular parking orbit, (i) the speed of the spacecraft in the
transfer ellipse at Mars, and (j) the necessary $\Delta v$ for insertion into the Mars parking
orbit.

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